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Heat flow through thermally anisotropic media: A numerical method and its application to an area of the southwest Scottish Highlands

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Abstract

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A numerical method is outlined for the solution of the conduction equation with source term for regions with thermal anisotropy. A number of general models are considered and temperatures, thermal gradients and heat flows are calculated to show the anisotropy effects. The model is applied to an area of the southwest Scottish Highlands to investigate the possible effect of thermal anisotropy on metamorphism. It is found that thermal anisotropy may have contributed to the metamorphism, but the most important effect is that of non-uniformity of thermal conductivity in the vertical direction. Deep local sources may also contribute to the conditions causing metamorphism.

Introduction

Some causes of local variation of heat flow from the Earth are the presence of local heat sources, variations in thermal conductivity, topography, underground water flow and convection, uplift and erosion, and past climate changes. Thermal anisotropy of the rocks is another factor which affects the nature of the heat flow near the surface of the Earth.

Munjal & Fatt (1966) have outlined a method for measuring thermal anisostropy in Earth materials and have observed such anisotropy in sandstone, bituminous coal, granite, limestone and rock salt. It has been recognized (Oxburgh et al. 1977) that such anisotropy may affect heat flow determinations. Also Borradaile (1976) suggested that widespread thermal anisotropy may have influenced the heat conduction pattern during the Caledonian metamorphism in the southwest Scottish Highlands.

The effects of thermal anisotropy are relevant to present day heat flow measurements as well as to better understanding the history of the local geology and the development of regional metamorphism.

Heat conduction in anisotropic solids has been discussed by Carslaw & Jaeger (1959), Nye (1960) and others, though apparently little work has been done to study local effects of thermal anisotropy in geological situations.

The present work describes a numerical method to calculate heat flow through zones where thermal anisotropy may occur and applies this approach to the area of the southwest Scottish Highlands considered by Borradaile (1976).

Numerical method

The conservation of energy equation for heat conduction with source term is:

$$\rho C \frac{\delta T}{\delta t} = \nabla . (K \nabla T) + H \tag{1}$$

where t = time
ϱ = density
C = specific heat
T = temperature
K = thermal conductivity
H = heat source term

For a two-dimensional model in Cartesian co-ordinates with co-ordinate axes in the x and z directions, and for the case of isotropic conductivity, (1) becomes

$$\frac{\delta T}{\delta t} = \alpha \frac{\delta^2 T}{\delta x^2} + \alpha \frac{\delta^2 T}{\delta z^2} + \beta \frac{\delta K}{\delta x} \frac{\delta T}{\delta x} + \tag{2}$$

$$+ \beta \frac{\delta K}{\delta z} \frac{\delta T}{\delta z} + \beta H$$

where $\alpha = \frac{K}{Cp}$ and $\beta = \frac{1}{Cp}$

If it is assumed that the conductivity is anisotropic, with principle conductivities K_x and K_z along the co-ordinate axes, (2) becomes

$$\frac{\delta T}{\delta t} = \alpha_X \frac{\delta^2 T}{\delta x^2} + \alpha_Z \frac{\delta^2 T}{\delta z^2} + \beta \frac{\delta K_X}{\delta x} \frac{\delta T}{\delta x} + \tag{3}$$

$$+ \beta \frac{\delta K_Z}{\delta z} \frac{\delta T}{\delta z} + \beta H$$

where $\alpha_X = \frac{K_X}{Cp}$ and $\alpha_Z = \frac{K_Z}{Cp}$

Equation (3) can be solved numerically in a similar manner to that used by Minear & Toksöz (1970). If we superimpose a mesh of grid points over the region of interest, which includes the directional variation of conductivity, and choose the grid system

$$x = n\Delta x, \; n = 1, 2, \ldots N$$
$$z = m\Delta z, \; m = 1, 2, \ldots M$$
$$t = p\Delta t, \; p = 1, 2, \ldots P$$

$$T^p_{n,m} = T^{p\Delta t}_{n\Delta x,\, m\Delta z},$$

we can write a finite difference expression for (3) in which temperature derivatives with respect to x are evaluated at (time) step p+1, and derivatives with respect to z at (time) step p. This results in an equation implicit in x:

$$\frac{T^{p+1}_{n,m} - T^p_{n,m}}{\Delta t} = \alpha_X \frac{T^{p+1}_{n+1,m} - 2T^{p+1}_{n,m} + T^{p+1}_{n-1,m}}{\Delta x^2} + \tag{4}$$

$$+ \beta \frac{K^p_{Xn+1,m} - K^p_{Xn-1,m}}{2\Delta x} \cdot \frac{T^{p+1}_{n+1,m} - T^{p+1}_{n-1,m}}{2\Delta x} +$$

$$+ \alpha_Z \frac{T^p_{n,m+1} - 2T^p_{n,m} + T^p_{n,m-1}}{\Delta z^2} +$$

$$+ \beta \frac{K^p_{Zn,m+1} - K^p_{Zn,m-1}}{2\Delta x} \cdot \frac{T^p_{n,m+1} - T^p_{n,m-1}}{2\Delta x} +$$

$$+ \beta H^p_{n,m}$$

This can be re-arranged to give

$$A_n T^{p+1}_{n-1,m} + B_n T^{p+1}_{n,m} + C_n T^{p+1}_{n+1,m} = D_n \tag{5}$$

where

$$A_n = \frac{-\alpha_X}{\Delta x^2} + \beta \frac{K^p_{Xn+1,m} - K^p_{Xn-1,m}}{4\Delta x^2}$$

$$B_n = \frac{2\alpha_X}{\Delta x^2} + \frac{1}{\Delta t}$$

$$C_n = \frac{-\alpha_X}{\Delta x^2} - \beta \cdot \frac{K^p_{Xn+1,m} - K^p_{Xn-1,m}}{4\Delta x^2}$$

$$D_n = \alpha_Z \cdot \frac{T^p_{n,m+1} - 2T^p_{n,m} + T^p_{n,m-1}}{\Delta z^2} + \frac{T^p_{n,m}}{\Delta t} +$$

$$+ \beta \cdot \frac{K^p_{Zn,m+1} - K^p_{Zn,m-1}}{2\Delta x} \cdot \frac{T^p_{n,m+1} - T^p_{n,m-1}}{2\Delta x} +$$

$$+ \beta H^p_{n,m}$$

An equation of the form (5) is obtained for each grid point in the m^{th} row, yielding a set of N simultaneous equations. The boundary conditions are included in the equations. This set of N equations yields the temperatures at (time) step p+1 at each grid point in the m^{th} row in terms of temperature, conductivity and heat generation known at the previous (time) step p.

Next, the finite difference expression for (3) in which derivatives of temperature with respect to z are evaluated at (time) step p+2 and derivatives with respect to x at step p+1 is written which gives an equation implicit in z. This last equation is like (4), and from it equations of the form of (5) are obtained for each grid point in the n^{th} column, yielding M simultaneous equations. The N sets of M simultaneous equations obtained at step p+2 may be solved to give the temperature at time p+2.

That is, in two steps, the solution is advanced first by solution of equation (4) which is implicit in x, and then by the solution of a similar equation implicit in Z. The technique incorporates the anisotropic nature of the thermal conductivity. Also, the method is capable of allowing for changes of conditions with time.

Modelling

General

Three models have been studied: two models with heat source regions embedded in layered media with different anisotropic characteristics and a third model which represents part of the southwest Scottish Highlands.

For all models, $\varrho = 2.76 \times 10^3$ kg m^{-3} C $= 1.3 \times 10^3$ J kg^{-1} K^{-1}. The mesh chosen for the calculations was 101 × 101 points, and vertical (K_z) and horizontal (K_x) thermal conductivity values were assigned to each grid point. For the layered models, the grid size was 50 m × 150 m and for the Scottish models it was 0.4 km square, which means that the sizes of the regions considered were 5 km × 15 km and 40 km × 40 km respectively.

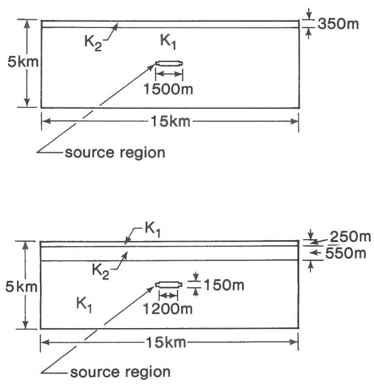
The upper boundaries of the models (the surface of the Earth) were maintained at 0°C. An initial linear temperature distribution with depth of 25°C km^{-1} was applied to the mesh, and the heat flux across the side boundaries of the models was taken as zero.

For the layered models, a constant heat flux of 50 mWm^{-2} into the bottom was applied. For the Scottish model, a constant heat flux of 82.5 mWm^{-2} into the bottom was found necessary to reasonably maintain the 25°C km^{-1} initial linear temperature distribution with depth for the thermal conductivities employed.

The layered models

The two layered models are shown in Figure 1, and the vertical (z) and horizontal (x) thermal conductivities for each region are given in the accompanying tables. Three thermal conductivity combinations were investigated for each model. Model 1 is a two-layered Earth with isotropic layers for model 1a, and with an anisotropic top layer for models 1b and 1c. Model 2 is a three-layered model. The top layer and the layer which contains the heat source and isotropic and of the same conductivity, but different from the intermediate layer. In model 2a the intermediate layer is isotropic, whereas in models 2b and 2c it is anisotropic.

In these layered models, the heat input from the source region was taken as 20 mWm^{-3}, and the time duration for the thermal fields to develop was taken as 158,000 years (5×10^{12} s). The results are



Model 1	a	b	c
K_{1x}	2.0	2.0	2.0
K_{1z}	2.0	2.0	2.0
K_{2x}	1.0	1.0	5.0
K_{2z}	1.0	5.0	1.0

Model 2	a	b	c
K_{1x}	2.0	2.0	2.0
K_{1z}	2.0	2.0	2.0
K_{2x}	1.0	1.0	5.0
K_{2z}	1.0	5.0	1.0

Fig. 1. The layered models with embedded sources regions.

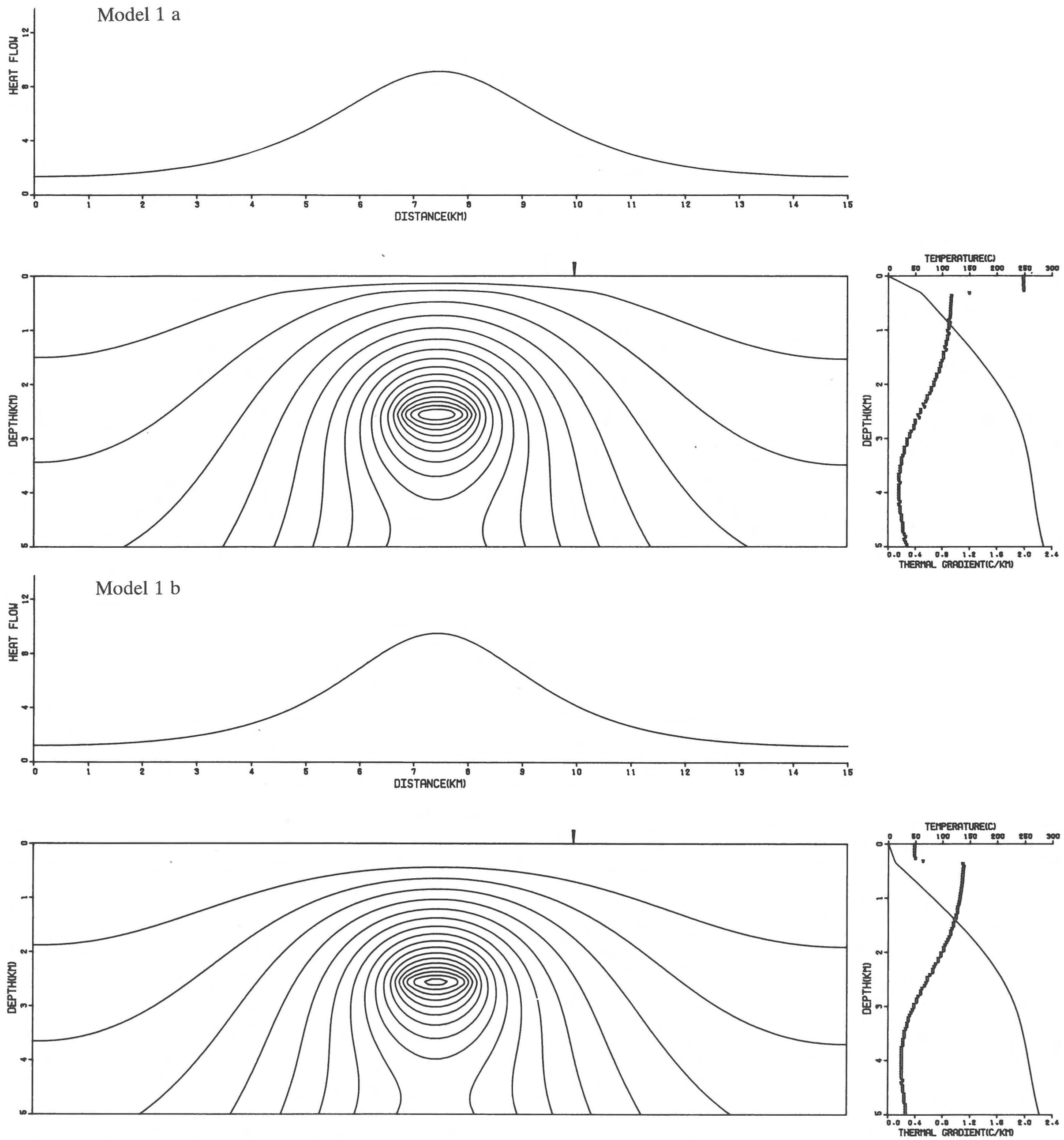


Fig. 2. Model 1. Vertical surface heat flow, temperature field and temperature and thermal gradient profiles at the position of the arrow. Heat flow is in units of 41.8 mWm^{-2}. The isotherms are at 50°C intervals starting at 0°C at the surface. The continuous line is the temperature profile, while the broken line is the thermal gradient.

shown in Figures 2 and 3. These illustrate the extent to which the embedded heat sources perturb the isotherms. For model 1, the effect of a large vertical thermal conductivity in the top layer (1b) produces a lower thermal gradient in that layer as compared to the isotropic case, and the isotherms are more deeply buried. Only small differences can be seen between the isotropic case (1a) and the case in which the horizontal conductivity is high in the top layer (1c).

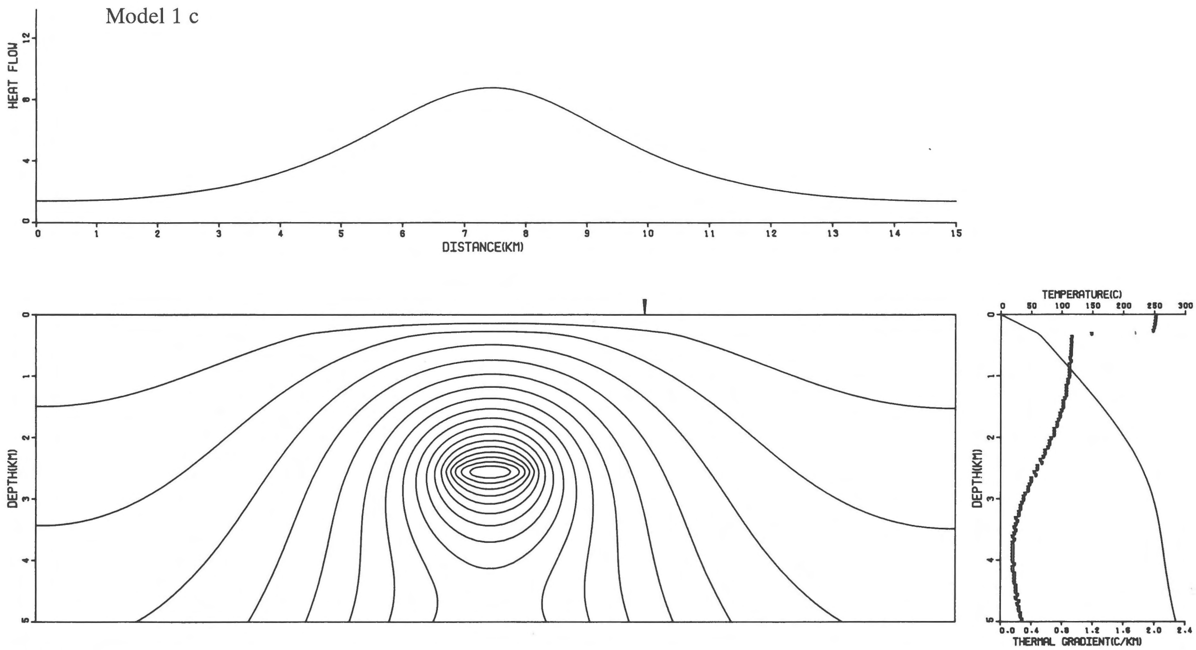


Fig. 2 (continued).

For the three-layer model (2), the surface heat flows, the twodimensional thermal fields, and the temperature-depth profiles differ for the isotropic and anisotropic cases, particularly for the case with higher vertical thermal conductivity. The larger vertical thermal conductivity in the intermediate layer strongly distorts the temperature field.

It is observed that when the vertical conductivity in the anisotropic layer is greater than the horizontal (Fig. 3b), the isotherms are considerably distorted and the vertical temperature gradient is substantially reduced for that interval. This leads to a temperature-depth profile that is distinctly different from the other two cases (Figs 3a, c). Nevertheless, the temperatures at depth approach each other in the three models. It is evident that the measurement of temperature gradients over short intervals cannot be used to predict temperatures at depth when anisotropic effects are present.

Although an increased thermal conductivity in the horizontal direction tends to distribute the heat laterally, its effect, at least for these cases, is negligible compared to that of an increase in vertical conductivity, which substantially perturbs the temperature field. It should be pointed out that such an increase in conductivity may be associated with thermal anisotropy as assumed here. However, it may also be associated with a larger thermal conductivity of the layer as a whole, without anisotropy, and similar results may occur.

The Scottish model

The Scottish model is shown in Figure 4. It is derived from the reconstruction of the Tay Nappe area in the southwest Scottish Highlands by Borradaile (1976), and is based on Figure 3b from that paper.

The geology and metamorphism of this area north of the Island of Arran was considered in detail by Atherton (1977). The prominent characteristic is a garnet zone that traces southwest to northeast through the area, Borradaile (1976), suggested that the location of the garnet zone may be controlled by the major structure, with the anisotropic character of the materials controlling heat conduction and metamorphism. He calculated mean thermal diffusivities and found a correlation between high mean diffusivity and the metamorphic zone. Borradaile (1976) therefore suggested that a large heat source may have existed below the present metamorphic zone which caused the iso-

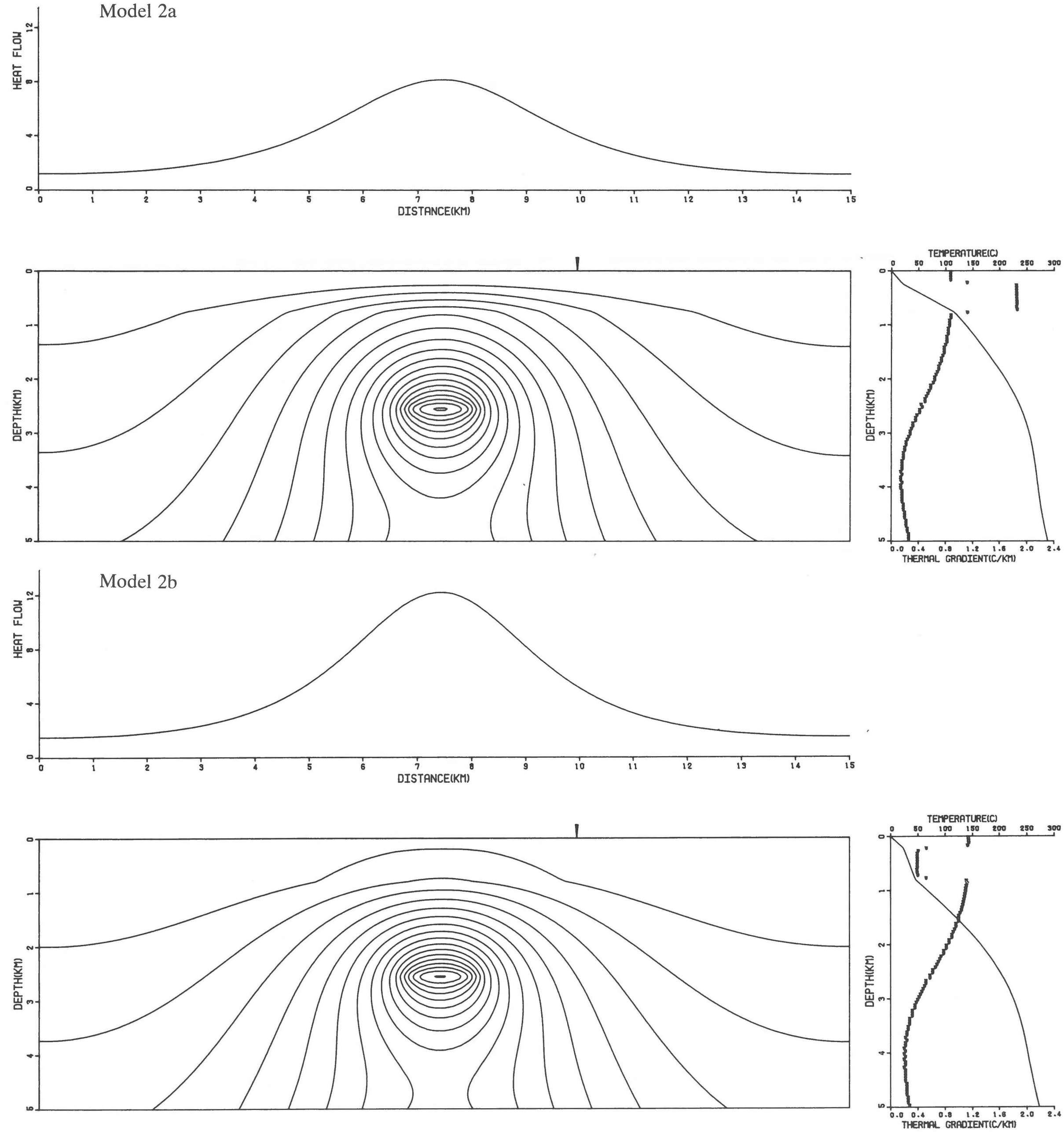


Fig. 3. As Figure 2, but for model 2.

thermal surfaces to be deflected upward along the steeper part of the Tay nappe with least thermal resistance in the vertical direction.

The purpose of the present model is to investigate the extent to which thermal anisotropy may have contributed to increased temperatures in the metamorphosed zone, and what influence a heat source at depth may have had on the thermal regime and the metamorphism.

To this end, eight models have been calculated, two without thermal anisotropy, and six anisotropic models, four of which had deep heat sources.

It was assumed that the present erosional level was situated 14 km below the surface of the Earth

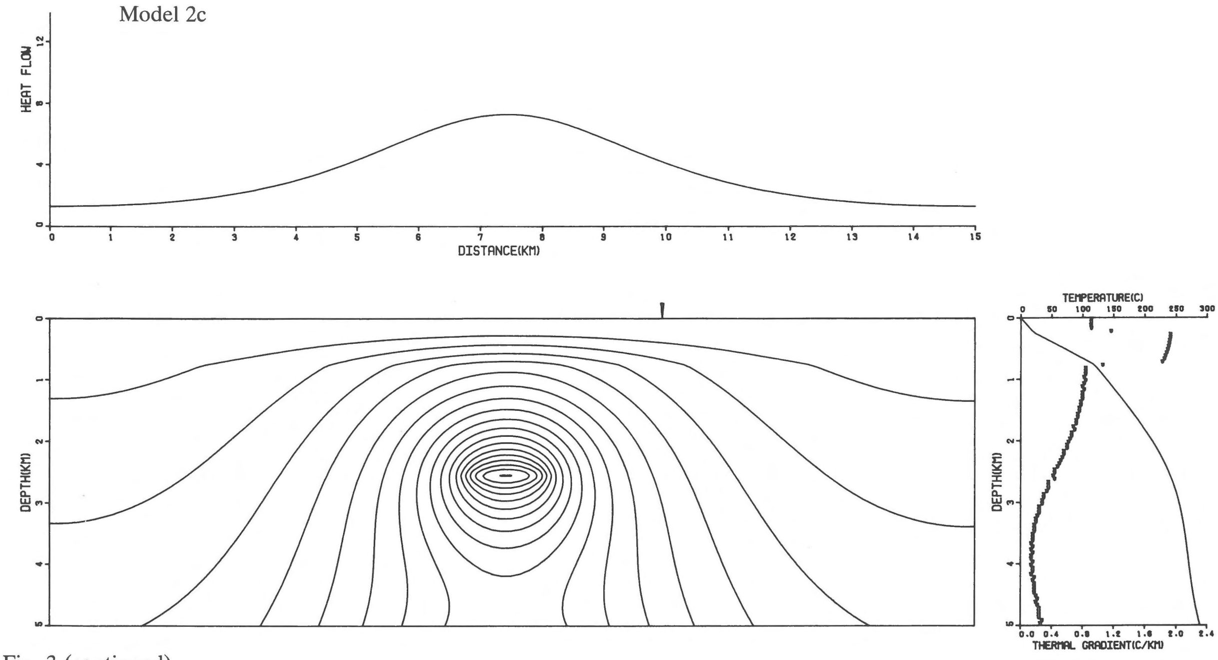


Fig. 3 (continued).

at the time of metamorphism. With a temperature gradient of 25°C km^{-1}, a temperature of 350°C is reached at that depth, thus remaining below the 400°C temperature reached in the garnet zone metamorphism.

Starting from initial horizontal isotherms, the thermal regime was allowed to develop for 15.9 million years (5×10^{14} s) for each case to see what conditions may have been appropriate to produce temperatures of the order of 400°C along the present erosional level.

The model was divided into six thermal conductivity zones corresponding tot specific rock types as indicated in Figure 4. The thermal conductivities applied to the zones are given in Table 1 and were derived from values appropriate for the types of rocks (Richardson & Powell 1976).

The result for case SCOT-4 as in Table 1 is given as an example in Figure 5. The temperature regime over the whole cross-section as well as the surface heat flow are given. Profiles of the effective thermal conductivities for the whole column, as well as for the sections of the column above and below the present erosion level are also shown. The effective thermal conductivities were calculated from:

$$K_{eff} = \frac{\Sigma L_i}{\Sigma\left(\frac{L_i}{K_i}\right)}$$

where L_i is the depth of each region of conductivity K_1 in the column.

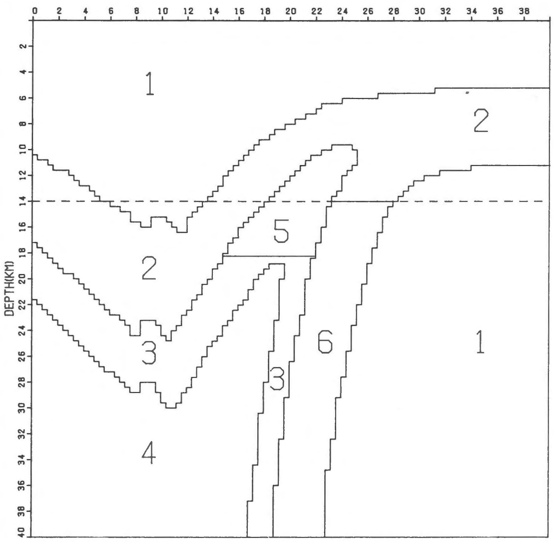


Fig. 4. The Scottish model. The numbers 1 to 6 mark thermal conductivity zones that correspond to specific rock types. The characteristic thermal conductivities for different models are given in Table 1. The dashed line represents the present erosion level.

Table 1. Thermal Conductivities for the Scottisch models.

		Thermal Conductivities (Wm-1 K-1)							
		Scot-1		Scot-2		Scot-3		Scot-4	
Zone	Rock Type	K_z	K_x	K_z	K_x	K_z	K_x	K_z	K_x
1.	Upper Dalradian	2.51	2.51	2.51	2.51	2.51	2.51	2.51	2.51
2.	Quartzite	3.34	3.34	3.34	3.34	3.34	3.34	3.34	3.34
3.	Phyllites (?)	3.85	3.85	3.85	3.72	3.85	3.85	3.85	2.72
4.	Mixed Quartzites + Shales	3.47	3.47	3.47	3.22	3.47	3.47	3.47	3.22
5.	Quartzites + Shales	3.47	3.47	–	–	–	–	–	–
5.	Tipped Phyllites	–	–	2.16	4.21	–	–	2.16	4.21
5.	Phyllites (?)	–	–	–	–	2.16	2.16	–	–
6.	Quartzite	3.34	3.34	3.34	3.34	3.34	3.34	–	–
6.	Quartzite (Vertical Schistosity)	–	–	–	–	–	–	3.85	2.75
	Heat Source (mWm-3)	–		–		–		–	

Figure 5 shows how the differences in thermal conductivities perturb the isotherms. The higher conductivities of the quartzites (2) and phyllites (3) conduct the heat upward and bring the isotherms nearer the surface, whereas the low conductivity of the upper Dalradian (1) tends to depress the isotherms. The heat is channelled by the higher conductivity zones. Although the temperature along the present erosional surface rises to just 400°C in the zone to the left, it remains below that temperature in the central zone for the time interval considered.

Figure 6 shows the results for the case SCOT-6B. The conductivities of this model are the same as those for Scot-4 (Figure 5), but a heat source has been included in the zone outlined by the dashed rectangle in the lower right side of the region. This heat source lies across a rather broad area, as was

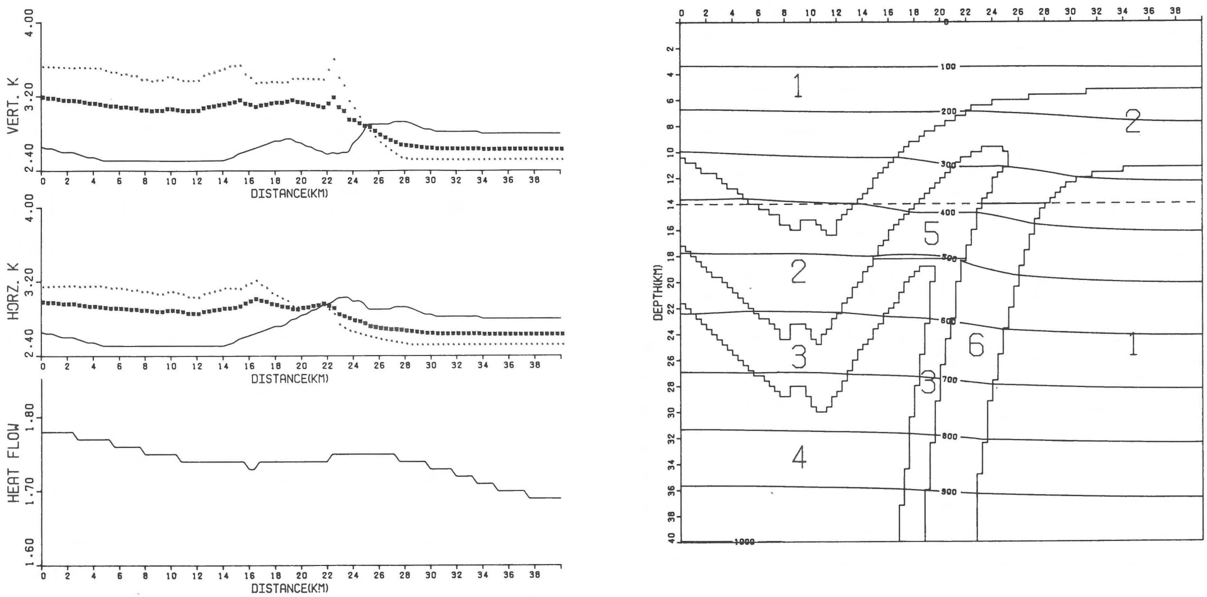


Fig. 5. Model Scot-4. Profiles of effective thermal conductivity in the vertical and horizontal directions (in Wm-1 K-1), and vertical heat flow at the surface (in units of 41.8 mWm-2) as well as the calculated thermal regime. Contour values are in °C. The effective thermal conductivities are calculated for the whole column (***), above the erosional surface (—) and below that surface (...).

Scot-5A		Scot-5B		Scot-6A		Scot-6B	
Kz	Kx	Kz	Kx	Kz	Kx	Kz	Kx
2.51	2.51	2.51	2.51	2.51	2.51	2.51	2.51
3.34	3.34	3.34	3.34	3.34	3.34	3.34	3.34
3.85	2.72	3.85	2.72	3.85	2.72	3.85	2.72
3.47	3.22	3.47	3.22	3.47	3.22	3.47	3.22
–	–	–	–	–	–	–	–
2.16	4.21	2.16	4.21	2.16	4.21	2.16	4.21
–	–	–	–	–	–	–	–
3.34	3.34	3.34	3.34	–	–	–	–
–	–	–	–	3.85	2.75	3.85	2.75
0.05		0.01		0.05		0.01	

suggested by Borradaile (1976), though the exact location is somewhat arbitrary. The input from the heat source is 0.01 mWm^{-3}. The effects on the isotherms and heat flow are dramatic. The temperatures along the erosional surface rise to 500°C in the central zone and high heat flow is produced in the zone above the heat source.

It has been found that the models with heat sources, the isothermal surfaces are locally deflected upward along the steeper part of the Tay Nappe as suggested by Borradaile (1976) and presence of the heat source has a great influence on the temperatures that occur.

Figure 7 compares the temperature profiles across the level of the present erosional surface after the time interval considered for all the models. It is seen that only when a heat source is included does the temperature in the region 18 to 26 km (the garnet zone) rise above 400°C. A heat input of 0.01 mWm^{-3} satisfies the conditions for metamorphism in the central zone, with temperatures that are not too high across the remainder of the region. Heat inputs into the source region of 0.05 mWm^{-3} produce temperatures that are unreasonably high across the whole region as can be seen from Figure 7,

Model 1 (no anisotropy) exhibits a fairly smooth temperature profile. In model 2, the effects of the different vertical conductivities are evident, and it is apparent that heat is transferred to the right in the central zone, though the direct effect of anisotropy is small, since a similar result is true for model 3 where only a change in thermal conduc-

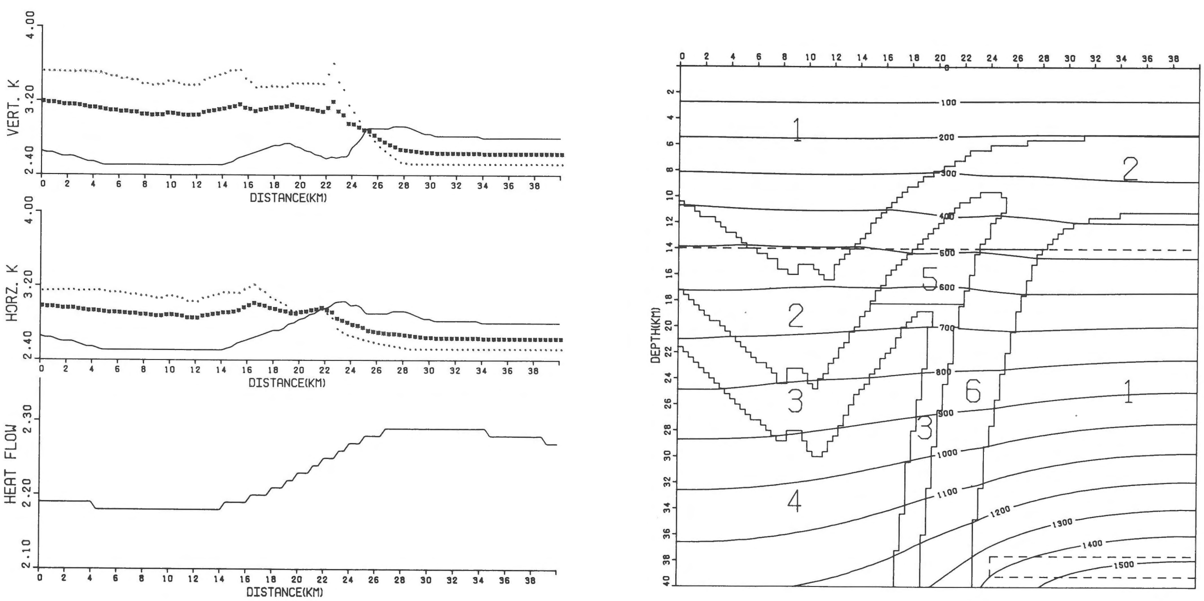


Fig. 6. As Figure 5, but for model Scot 6b. The conductivities for this model are the same as for Scot-4.

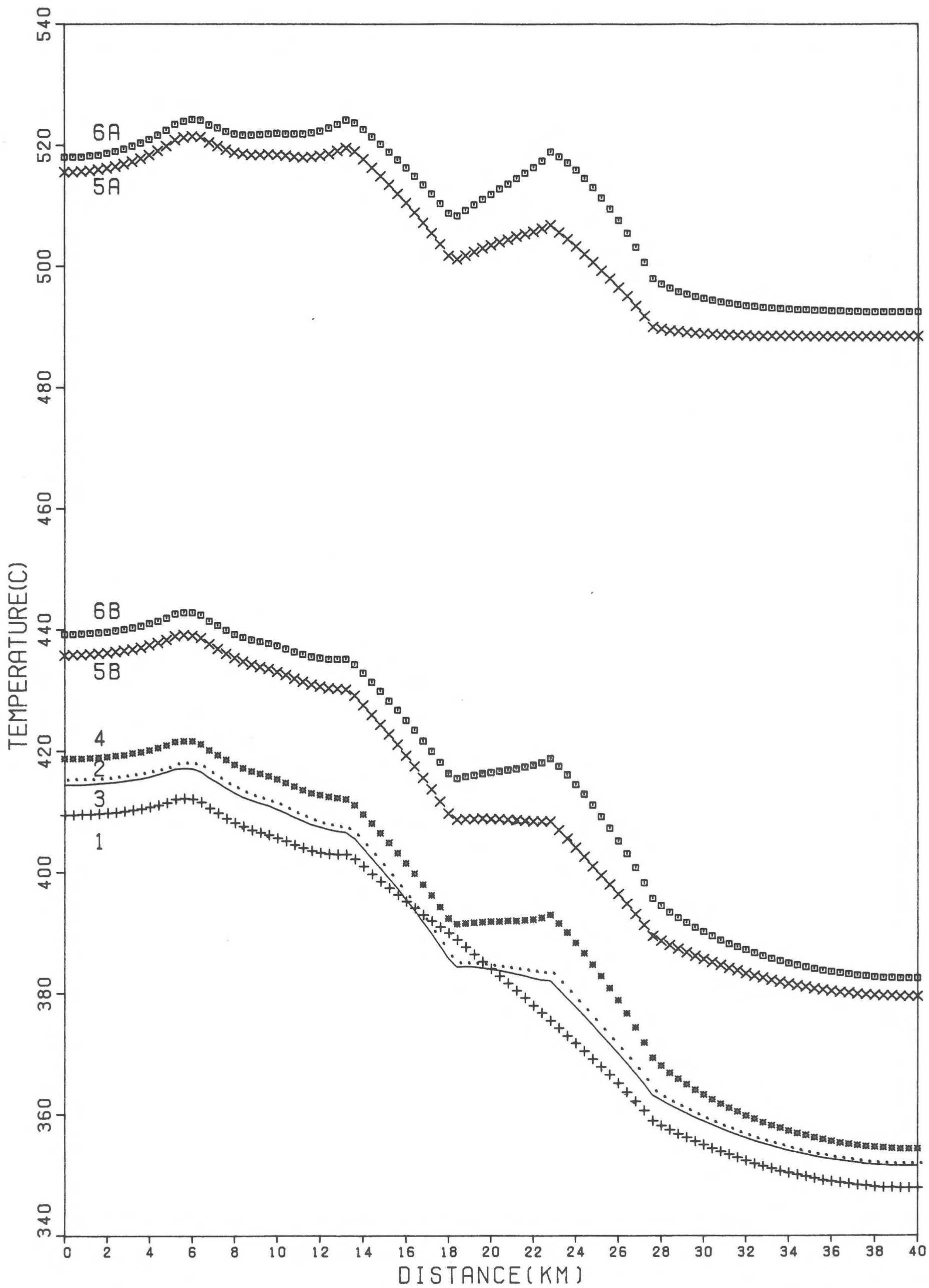


Fig. 7. Profiles of temperatures across the present erosional surface for the 8 models.

tivity occurs with no anisotropy. Model 4 shows the effect of the increase in vertical conductivity in zone 6, the quartzite with vertical schistosity, and the temperature is substantially raised in the central zone, but still lies below 400°C.

Conclusions

It has been found that thermal anisotropy perturbs the isotherms, thermal gradients, temperature profiles and heat flow. This is particularly true when the anisotropy is of increased thermal conductivity in the vertical direction. When the anisotropy is of increased conductivity in the horizontal direction, there is little effect.

The main disturbance to heat flow in actual cases, however, appears to be channeling of heat by higher conductivity zones, whether they are associated with the bulk thermal conductivity of the rock or thermal anisotropy. The perturbation to the thermal field associated with such channelling can be greater than any affect associated with anisotropy of the order of that investigated here.

For the models considered here for the Tay Nappe area, it appears that anisotropy alone cannot have raised the temperatures above 400°C at the erosional level to produce metamorphism in the garnet zone. However, if a heat source region is included, this effect can be obtained, and higher heat flow values occur southward of the region.

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