

A THEORY OF COMPOUND DECISIONS ON MILL-DUMP CUTOFF GRADES¹

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ABSTRACT

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For a large-scale mine and concentrator, working a low-grade orebody by open-pit methods, the mill-dump cutoff grade decision is of crucial economic importance. Lane has provided a sound theoretical basis for this decision-making. His approach to the problem is extended to allow for the effect of taxation, and to include deliberate decisions concerning the concentrator operations that must be taken in conjunction with the cutoff grade selection for optimal overall results.

The cutoff grade decision is thus seen as a compound decision upon which the government is able to exert influence through its tax formula.

It is assumed that a grade distribution is available for selective mining units, as well as mathematical expressions that describe the processes of comminution, liberation of valuable mineral from gangue, and mineral recovery by separation from gangue. Typically, the operation may be thought of as the mining and processing of the massive chalcopyrite-bornite zone of a porphyry orebody.

Apart from its operational use, the theory enables the identification of taxation schemes that enhance overall mineral recovery and government revenue, without affecting the economic feasibility point of the operator. Two simple tax formulae that achieve this are discussed.

REVIEW AND PROBLEM-SETTING

The economic attractiveness of an orebody with disseminated mineral is governed by a number of quality parameters:

- (1) grade, determining the amount of ore to be processed for unit mineral recovery;
- (2) ease of comminution (expressed, for example, in terms of the work index W) in producing concentrator feed of some characteristic grindsize η , determining the energy expenditure per unit ore for that grindsize;
- (3) ease of liberation of mineral from gangue, governed by liberation parameters, such as, for example, the mean effective mineral grain size l_m , (SCHAAP, 1979), which sets the grindsize;
- (4) ease of separation and recovery of liberated mineral from gangue, governed by parameters r_m (including native hydrophobicity if the flotation process is used), which determine the unit cost of concentration.

All of these parameters may be expected to be variable within the same ore body: they will exhibit a distribution of values with at least some regional correlation. All of these

parameters should enter into the ore body assessment procedures, in so far as their identification and quantification permits, and all of these parameters should have a bearing on the ore-waste classification in open-pit mining.

The purpose of this paper is to present a cutoff grade theory that will enable to take these quality parameters into account, via the mathematical models of processing, and that, at the same time, allows for the effect of taxation. It is strongly based on original approach by LANE (1964), which is now discussed and re-set in a form coherent with the further development.

A rational cutoff grade decision could be based on the objective of the operator to maximise the net present value of the remaining life of the project at any stage in its life. Actual behaviour of operating companies is, however, often not entirely consistent with this objective. Ignoring such possible, perhaps unintentional, inconsistency, an objective of this nature, concerned with an optimal trajectory, invites the use of optimal control theory or dynamic programming. The latter is less demanding of the continuity of mathematical descriptions of processes and their inter-relations and thus offers advantages especially for problems of low dimensionality: the dynamic programming formulation will be used.

The time required to work the remainder of the ore body

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at any stage of the project is not known until the problem has been solved (if it is known, because it is set, extraction is likely to be sub-optimal: RUDENNO, 1979). The extent of the ore body in so far as it affects current decision making is known. A *stage* of the project is thus best characterised by the volume of the ore body still in place Q ; this volume may be expressed as a discrete number of parcels of material. Let $V(Q)$ be its net present value to the operator and Q_m the parcel of material next to be worked for a certain net return. This stage return will obviously be some function of mineral price and costs, but also of quality characteristics of the material - grade, work index, effective mean mineral grain size, native hydrophobicity- as well as the taxation liability. The operator can exercise some control over the return function through his decision variables k . This decision vector must be selected from a set of permissible decision vectors K (unlike in other optimisation techniques, such constraints are actually helpful in dynamic programming).

If two conditions are met the problem can be cast into a convenient recurrence relation. These conditions are the separability and optimality conditions (e.g. HASTINGS, 1973). In the current context both conditions *can* be violated if insufficient care is taken. For example, if dump leaching is practised on part or all of the material not sent to the mill, the return on the relevant actions, included in K , is not yet known by the time operating decisions have to be made for the next parcel of material: the separability condition is not met. This is not too serious if a reasonable prediction of the net return, appropriately discounted by the operator's discount rate r , can be made and thus included in the present stage return. Another problem could arise if some form of progressive resource rent tax were levied based on the difference between previous total net operator return and some target value set by the government. In this case it is possible that, with regards to total net discounted operator return over n parcels of material, the use of the optimal policy over $(n-1)$ parcels followed by a maximum stage return leads to a sub-optimal result, as the optimality condition is violated.

Assuming that the two conditions are met, and denoting the stage return by $r(Q_m, k)$, with T representing the time required to work Q_m , the problem may be stated by the following backward recurrence relation.

Find for all stages:

$$V(Q) = \max_{k \in K} \left\{ r(Q_m, k) + \frac{V(Q-Q_m)}{(1+r)^T} \right\} \quad (1)$$

For $rT \ll 1$ this relation may be written as:

$$V(Q) = \max_{k \in K} \{ r(Q_m, k) + V(Q-Q_m) - rTV(Q-Q_m) \} \quad (2)$$

or:

$$V(Q) - V(Q-Q_m) = \max_{k \in K} \{ r(Q_m, k) - rTV(Q-Q_m) \} \quad (3)$$

The left hand side of equation (3) clearly expresses the optimal net contribution to the operator's present project value that can be made by working parcel Q_m . Current stage return must exceed $rTV(Q-Q_m)$ to make a net contribution. The operator is thus under pressure to reduce the time spent on working the parcel Q_m ; this pressure diminishes as the value of the quantity unmined material decreases. This notion, already present in the work of GRAY (1914) and in the definition of mine rent by CASSEL (1932), appears to have been first clearly stated as the principle that should govern mill-dump cutoff grades by LANE (1964). TAYLOR (1972), in a thorough review of the subject, found no earlier consistent treatment. BLACKWELL (1971) showed this first in a manner similar to the above.

Relation (2) is to be solved working backwards, parcel by parcel, using best possible price and cost estimates and ore characteristics. During planning it must be solved right through several times, as the operator has to select the scale of operations and the sequencing of pit developments. This is because the scale of operations decides to a large extent the cost functions, and may also affect the operator's cost of capital and thus his discount rate r . RUDENNO (1979) considered these aspects. The only decision variable recognised in the literature is the cutoff grade itself. Through this variable the operator decides the quantity of ore in each parcel and thus the time spent on working it for a fixed concentrator capacity, and the flow of revenue for a fixed concentrator recovery. Together with the various cost functions the stage return is then calculated. BLACKWELL (1971) provided an illustration of this.

During operations, as projected prices and costs are re-estimated with more confidence, the value iteration (2) will be repeated to check operating strategy and re-estimate $V(Q-Q_m)$, although now the scale of operations is fixed, at least in the short term, except for some innate flexibility in mining capacity once pit access restrictions no longer exist. There is one other difference compared to planning: while working the current parcel Q_m the operator faces real prices and costs, not projected ones. Because of the volatility of metal prices, cost-price relations may differ greatly from the expected ones. Relation (3) must be modified to take this into account, as BLACKWELL (1971) did, and LANE (1979) explained in more detail. In conducting the present operations on Q_m the realisation of $V(Q-Q_m)$ is deferred a time T at an opportunity cost of $rTV(Q-Q_m)$ according to (3). If prices are, relatively, high at present there is an extra component of opportunity cost, for, even if projected prices and costs beyond T are right, the bringing forward of the realisation of the value of $(Q-Q_m)$ would result in $V(Q-Q_m)$ increasing at a rate:

$$-\frac{\delta V(Q-Q_m)}{\delta T}$$

In spending time T on the current parcel Q_m a benefit of:

$-T \frac{\delta V}{\delta T}$ is foregone; the fact that this can not entirely be helped is immaterial. The operator is thus under extra pressure to reduce T . Similarly, if prices are currently low, the opportunity cost of deferring further extraction is reduced. The modified recurrence relation is thus:

$$V(Q) - V(Q-Q_m) = \max_{k \in K} \{r(Q_m, k) + T \frac{\delta V(Q-Q_m)}{\delta T} - rTV(Q-Q_m)\}. \quad (4)$$

The dynamic programming formulation of the problem has thus produced a finite series of stage-optimisation problems in which the stage return function is prominent. Of course, the return function may be allowed to vary from stage to stage – and the stages need not be equal in size – but the correctness of its expression is obviously of the greatest essence.

The appropriate stage return is the *net* cash flow to the operator, not the *before tax* cash flow adopted by Lane and Blackwell in their publications. Furthermore, in considering cutoff grade as the only decision variable, both Lane and Blackwell have assumed constant concentrator capacity and recovery, seemingly exogenously determined. These are *not* exogenous variables: they are subject to deliberate operator decision-making in dependence of the ore quality parameters prevailing in the parcel Q_m . A lower work index permits a greater capacity for the same grind size, a lower mean effective grain size needs a finer grind size, and thus lower concentrator capacity, for the same recovery. The cutoff grade decision should, in general, not be a single decision; k has more than one component.

DEVELOPMENT OF THE THEORY

The general expression for the stage return needs the following definitions:

$Q_c(Q_m, g_c)$ the amount of material, in m^3 , classified as ore (sent to the mill) within the parcel Q_m , governed by the cutoff grade of mining units g_c ;

\bar{g} the average grade of all mining units of Q_m , classified as ore, expressed as a volume fraction if the proposed mineral liberation model is to be used;

m direct cost of mining per m^3 of material, i.e. drilling, blasting, loading, trucking, sampling. Trucking costs are taken as the same for ore and waste; if there is a difference it is allowed for in the cost of concentrating ore;

c variable cost of concentrating per m^3 ore; this will exclude power and labour costs since these have in operating mills the character of fixed costs. Conditioning costs and the variable cost of tailing disposal are major components with, possibly, a positive or negative allowance for the difference in trucking cost of ore and waste. It may be necessary to split this cost into a component c_0 per m^3 ore

and a component c_1 per m^3 mineral recovered; the latter expressing, for example, the cost of reagent adsorbed by the mineral recovered;

f fixed costs per unit time, only real costs, e.g. maintenance of roads and buildings and other infrastructure not provided by the government, administration, concentrator power and labour, that are independent of the precise level of production;

s the value of 1 m^3 mineral contained in the concentrate, net of drying, storage, shipping and marketing costs;

y fractional recovery of mineral in the concentrator;

r operator's discount rate;

F concentrator capacity at work index W , and grind size characteristic η , expressed as the feed rate to the mill, m^3 of ore per unit time;

M mining capacity in m^3 of material shifted per unit time;

S market capacity in m^3 of mineral in concentrate per unit time;

θ government tax formula applicable to parcel Q_m .

The stage return, as net cash flow to the operator, is:

$$r(Q_m, k) = (y\bar{g}s - c)Q_c(Q_m, g_c) - mQ_m - fT - \theta$$

The stage optimisation problem is thus:

$$\max_{k \in K} \{ (y\bar{g}s - c)Q_c(Q_m, g_c) - mQ_m - \theta - fT - rTV(Q-Q_m) + T \frac{\delta V(Q-Q_m)}{\delta T} \}$$

subject to the constraints:

$$\frac{Q_m}{M} \leq T, \quad \frac{Q_c}{F} \leq T \quad \text{and} \quad \frac{S}{y\bar{g}Q_c} \leq T. \quad (5)$$

Not all constraints in (5) are equally hard. The least flexible capacity constraint in mature workings will be that of the concentrator: mining and market capacities are sought to match the (optimum) concentrator capacity as closely as possible. From the planning point of view it is thus desirable to ensure that best concentrator feed rate and concentrate production rate do not vary wildly from one programmed quantity Q_m to the next.

In extremely poor market conditions it may only be possible to sell a small part of the concentrate that could be produced. Part of the concentrator, which consists generally of a number of parallel circuits, could then be laid idle with a consequent reduction in power costs, reflected in f . The best course of action in these circumstances depends so much on specific local conditions including some intangible political and personnel motivations, that it is best left out of a general theory of cutoff grades. It will therefore be assumed that market conditions are fully reflected by price-cost relations as accounted for by the term:

$$T \frac{\delta V(Q-Q_m)}{\delta T}$$

in the objective of (5). The last constraint will be considered no further.

Furthermore, a gross mismatch between mining capacity and concentrator capacity can not be tolerated by the operator, although a perfect match at all times may be unattainable. Two instances where a serious mismatch may temporarily arise are when: (1) early in the life of the project full concentrator capacity has come on stream before the pit has developed the necessary access for full scale production, and (2) later, a push-back cuts through lateral deposit boundaries thus draining mining equipment away for overburden removal.

For the part of the ore body that is the concern in this work, the first two capacity constraints may be combined, since the concentrator can process no more ore than the mine produces, and vice versa, other than momentarily:

$$F = \frac{Q_c}{Q_m} M$$

The time required to work the current increment Q_m is then equal to:

$$T = \frac{Q_c}{F}$$

Writing: $v = V(Q) - V(Q - Q_m)$, and suppressing the dependence of Q_c on Q_m and g_c , and of V on $(Q - Q_m)$ in the notation for brevity, this discussion modifies the problem to:

$$\max_{k \in K} v = \left\{ y s \bar{g} - c - \frac{f + rV - \frac{\delta V}{\delta T}}{F} \right\} Q_c - m Q_m - \Theta$$

$$\text{subject to: } F = M \frac{Q_c}{Q_m} \quad (6)$$

The variables in (6) need close inspection to determine which are controllable and which are uncontrollable. Controllable means in this context, subject to decision-making by the operator, since he makes, axiomatically, all operating decisions. Thus, Θ , a control variable for the government, is *not* a control variable in (6); rather, the government must devise a formula for Θ such that if the operator solves (6), he fulfills the government's objective concurrently with his own. Further, for a given increment of material Q_m the unit mining cost m is set, the fixed cost per unit time f and the discount rate r are constants; V , as the optimal net value of the remainder of the deposit has a specific value at this stage; $\delta V / \delta T$ is determined by exogenous economic conditions, as is the unit net mineral value s . These are thus variables over which the operator has no policy control at all. The unit concentrating cost c poses a problem: part of it will be proportional to the amount of ore treated, but often a part of it will depend on the amount of mineral treated, for example, some reagent costs. As an approximation the concentrating cost is decomposed in a component c_0 per m^3 ore treated and a cost c_1 per m^3 mineral recovered, which may be looked upon as the cost of reagents consumed by the mineral. The latter component is thus a decision variable for the operator, who balances reagent costs with recovery benefits. The mean

grade of ore \bar{g} depends on the grade distribution of mining units and the cutoff grade g_c selected by the operator, which also determines Q_c . In fact, if $f(g)$ represents the probability density function of the mean grade of selective mining units, it follows that:

$$\bar{g} = \frac{\int_{g_c}^{\infty} g f(g) dg}{\int_{g_c}^{\infty} f(g) dg} \quad (7)$$

$$\text{and } Q_c = Q_M \int_{g_c}^{\infty} f(g) dg. \quad (8)$$

The concentrator feed rate F depends on the work index W (or other grindability parameter(s), if preferred) of the ore, as the second quality parameter, but also on the fineness of grind desired, as expressed by the characteristic size η . This fineness of grind – controlled by classifier operating variables, e.g. pulp density, in turn – thus qualifies as the third decision variable for the operator, via:

$$F = F(W, \eta). \quad (9)$$

The concentrator recovery depends on the ease of mineral liberation, the third ore quality parameter, represented by l_m ; on the fineness of grind η ; on reagent use c_1 , determined largely by a fourth ore quality parameter related to ease of recovery, e.g. native hydrophobicity r_m ; and also on the feed rate as it affects the time constants of recovery equipment.

$$y = y(l_m, \eta, c_1, r_m, F(W, \eta)).$$

Since for a given ore and plant there is a one-to-one relation between feed rate and grind size this may also be written as:

$$y = y(l_m, W, F, c_1). \quad (10)$$

In practice recovery seems also to depend on head grade \bar{g} , but this is taken to occur via a correlation between grade and liberation parameters.

In summary, g_c , F , and c_1 thus serve as the operator's decision variables. For a given parcel Q_m results will be parametrically dependent upon the effective means of the ore quality variables l_m , W , and r_m ; during the pit planning stages these quality variables, with grade, must play a part in sequencing pit sections.

The stage problem to be solved can thus be fully specified in the following form, where it is understood that the decision variables all lie within their natural limits:

$$\begin{aligned} \text{maximise: } v = \{ & Q_m \left\{ y(s - c_1) \int_{g_c}^{\infty} g f(g) dg - \right. \\ & \left. - \left\{ c_0 + \frac{f + rV - \frac{\delta V}{\delta T}}{F} \right\} \int_{g_c}^{\infty} f(g) dg - m \right\} - \Theta \end{aligned}$$

subject to: $F = M \int_{g_c}^{\infty} f(g)dg.$ (11)

Each stage in the value iteration procedure to determine the cutoff grade policy for the remainder of the project is itself a non-linear optimisation problem, that may be solved using certain, now well-established principles of mathematical programming (e.g. LUENBERGER, 1973).

Using the Lagrange multiplier method, the Lagrangian of (11) is defined by:

$$L(g_c, F, c_1, \lambda) = v - \lambda(F - M \int_{g_c}^{\infty} f(g)dg). \quad (12)$$

The first order or necessary conditions for optimality are the Kuhn-Tucker conditions:

$$\frac{\partial L}{\partial g_c} = 0, \quad (13)$$

$$\frac{\partial L}{\partial F} = 0, \quad (14)$$

$$\frac{\partial L}{\partial c_1} = 0 \quad (15)$$

$$\text{and } \lambda(F - M \int_{g_c}^{\infty} f(g)dg) = 0. \quad (16)$$

With regard to the Lagrange multiplier λ , introduced in definition (12), it is noted that its value at optimal values for g_c , F , and c_1 , will be negative if mining capacity is greater than the ideal match, positive if it is less. In the former case it has the interpretation of the net benefit to be gained by an extra m^3 per unit time of concentrating capacity, in the latter case of the net benefit to be gained by an extra unit of ore production capacity. Mine production rate should thus be planned in such a way that the value of λ is kept as close to zero as possible. If the multiplier is slightly negative, the slight excess in mining capacity could perhaps be best used in producing some bufferstock for use in periods when it would otherwise be slightly positive, rather than enforcing a match between mine and concentrator. This point will be raised again after expanding the conditions (13) to (15). From (13):

$$Q_m \left\{ -y(s-c_1)g_c f(g_c) + \left\{ c_0 + \frac{f+rV-\frac{\delta V}{\delta T}}{F} \right\} f(g_c) \right\} - \frac{\partial \Theta}{\partial g_c} - \lambda M f(g_c) = 0;$$

From (14):

$$Q_m \left\{ (s-c_1) \frac{\delta y}{\delta F} \int_{g_c}^{\infty} g f(g)dg + \frac{f+rV-\frac{\delta V}{\delta T}}{F^2} \int_{g_c}^{\infty} f(g)dg \right\} - \frac{\partial \Theta}{\delta F} - \lambda = 0;$$

from 15:

$$Q_m \left\{ (s-c_1) \frac{\delta y}{\delta c_1} \int_{g_c}^{\infty} g f(g)dg - y \int_{g_c}^{\infty} g f(g)dg \right\} - \frac{\partial \Theta}{\delta c_1} = 0.$$

Some simplification and rearrangements lead to the following set of equations that must be satisfied simultaneously, if the cutoff grade, concentrator feed rate and reagent use are optimal:

$$g_c = \frac{c_0}{y(s-c_1)} + \frac{f+rV-\frac{\delta V}{\delta T}}{Fy(s-c_1)} - \frac{\lambda M}{y(s-c_1)Q_m} - \frac{1}{(s-c_1)yQ_m f(g_c)} \frac{\partial \Theta}{\partial g_c} \quad (17)$$

$$\frac{\delta y}{\delta F} = \frac{1}{\int_{g_c}^{\infty} g f(g)dg} \left\{ \frac{\lambda}{Q_m (s-c_1)} + \frac{\partial \Theta}{\delta F} \frac{f+rV-\frac{\delta V}{\delta T}}{(s-c_1)F^2} \int_{g_c}^{\infty} f(g)dg \right\} \quad (18)$$

$$\frac{\delta y}{\delta c_1} = \frac{y}{(s-c_1)} + \frac{1}{Q_m(s-c_1)} \int_{g_c}^{\infty} g f(g)dg \frac{\partial \Theta}{\delta c_1}. \quad (19)$$

The cutoff grade equation (17) is essentially identical to the formula used by Lane and Blackwell, for the case where the concentrator capacity decides production rate, with the addition of two terms; the first of these describes the effect of forcing a match between mine and concentrator capacities, the second the effect of taxation. The other major difference is that feed rate, recovery, and reagent use are determined by the simultaneous equations (18) and (19), which in turn depend on the cutoff grade. Equation (18) shows that concentrator feed rate can be forced up to a certain point, where the resulting loss in recovery balances the savings in the cost of time, depending on the cutoff grade. Equation (19) shows the cutoff point on conditioning expenses c_1 : reagents should be added up to the point where the marginal improvement in recovery equals the right hand side of (19).

A few immediate observations can be made from these equations:

(1) Equation (17) shows that if $\delta V/\delta T$ is positive, i.e. when current price-cost relations are worse than expected, there is

a tendency for g_c to fall, with a consequent improvement in mineral recovery from the pit. At the same time there is a tendency for $\delta y/\delta F$ to be less negative by equation (18), with a consequent lower feed rate and better concentrator recovery as well. Less material need then be mined per unit time, resulting in possible idle mining capacity. If $\delta V/\delta T$ is negative the opposite will occur. To make use of short-term price fluctuations there is something to be said for maintaining two buffer stocks, one with good grades to draw on in high-price periods, one with marginal grades for use when prices are depressed; the temporary excess in mine capacity could be used for this purpose. This is in agreement with views expressed by LANE (1979) and TAYLOR (1972).

(2) It is seen that cutoff grade declines and concentrator recovery improves as the net value of the remainder of the deposit diminishes.

(3) If the mine capacity is too small to ideally match the concentrator production rate, λ is positive and the cutoff grade will be less, concentrator recovery higher, than with an ideal match. The opposite is true if the mine capacity in actual use is greater than with an ideal match.

(4) Mining costs, and by implication overburden removal costs, do not affect the cutoff grade decisions. The ultimate stripping ratio can thus be decided upon after the optimal revenue has been calculated from the cutoff grade and recovery computations according to equations (17) to (19). It must be noted, however, that the very actions of removing overburden can influence the cutoff grade via the Lagrange multiplier, if they result in under capacity of the mine through equipment sharing.

If the solution to (11) is obtained by direct iterative solution of equations (17) to (19), the optimality of the candidate solution vector (g_c, F, c_1) should be checked with the second-order or sufficiency conditions. It is clear from the objective of (11) that v attains an absolute minimum if all material is dumped, and a weaker minimum if all material is treated. If there were to exist another local minimum between these natural extremes, there would have to be at least three stationary points of v ; this is not likely. Thus, except in the case of severe overgrinding rendering mineral unfloatable, $\delta y/\delta F$ will always be negative in the natural operating range and $\delta y/\delta c_1$ will always be positive: no operator will throw chemicals away; there is also monotonicity in the grade integrals. There are, therefore, no indications that v should behave erratically within the natural operating limits. Although this argument in no way constitutes a proof that the stationary point given by equations (17) to (19) is a maximum, it provides some comfort. SCHAAP (1981) gives further detail on the sufficiency conditions. Alternative solution methods may, however, be developed if the operating models are very complex (equations (9) and (10)), based on, for example,

gradient methods or penalty functions (LUENBERGER, 1973).

It could be advantageous to solve equations (17) to (19) by initially setting the Lagrange multiplier to zero; the resulting cutoff grade and feed rate determine the ideally matched mining rate. If this mining rate cannot be exactly planned for, the nearest acceptable rate can be taken for re-solution of equations (17) to (19). If there is still freedom to choose the parcel Q_m , there is an alternative of including material that permits higher or lower concentrating rates, as required: this is strictly a pit-planning activity, not an ad-hoc operating decision.

ENHANCED RECOVERY THROUGH SPECIAL TAXATION

Governments endeavour to collect greater shares of the economic rents yielded by mining operations than that provided by a flat rate of company taxation. Often, these additional forms of taxation are imposed without regard to the profitability of the operation, for example, royalty and rental payments. From an inspection of equations (17) to (19) the following conclusions may be drawn immediately:

(1) An extra tax burden in the form of a rental payment is equivalent to an addition in the fixed costs f . By equations (17) and (18), both mineral recovery in the mine and in the concentrator will then be reduced, although by equation (19) extra use of reagent will to some extent counter-act the forced feed rate dictated by equation (18).

(2) A royalty levied on mineral contained in concentrate has the effect of lowering the net unit value of mineral to the operator s . Equations (17) to (19) show that mine and mill recovery will fall, without compensating side effects.

(3) A royalty based on material shifted, whether ore or waste, is equivalent to an increase in the unit mining cost m . This form of royalty is thus neutral to overall recovery, since mining cost does not enter equations (17) to (19).

The above three points are first-order effects; a secondary effect of all forms of extra taxation is the lowering of the net value to the operator of ore still in the ground, V . This reduces the opportunity cost of time spent on working the current increment Q_m , and thus has a beneficial effect on overall recovery that could partly or wholly off-set the harmful effects in points (1) and (2), provided V can be reduced sufficiently without losing the operator's interest.

Nevertheless, these indiscriminate forms of royalty or rental payments, not having any basis or justification in economic theory, cannot be rationally recommended.

Rather, if a royalty is extracted, it should be based on the material's capability to earn it, in agreement with economic theory. Accordingly the royalty could be levied on the amount of mineral in ore sent to the mill that is in excess of the economic break-even quantity. To recoup the unit cost of concentrating c_0 and the unit cost of time $(f+rV)$, the margin-

al mining unit must have a grade of:

$$\frac{c_0}{y(s-c_1)} + \frac{f+rV}{Fy(s-c_1)}.$$

The break-even grade g_r is then obtained by demanding that the cost of mining, including that of waste, be recovered as well:

$$g_r = \frac{c_0 + \frac{f+rV}{F_r}}{y_r(s-c_1)_r} + \int \frac{m}{\frac{c_0 + \frac{f+rV}{F_r}}{y_r(s-c_1)_r}} dg$$

If the average grade of all ore sent to the concentrator is g_r , no loss is incurred. The first term can be calculated from equations (17) to (19) on the zero-tax assumption; it then fixes the second term. For a parcel of given material g_r is a constant (if W , l_m , and r_m , or their effective means, are constant within this quantity of material). Let now:

$$\begin{aligned} \theta &= tQ_c(\bar{g}-g_r) = t(\bar{g}-g_r)Q_m \int_{g_c}^{\infty} f(g)dg \\ &= Q_m t \int_{g_c}^{\infty} gf(g)dg - Q_m t g_r \int_{g_c}^{\infty} f(g)dg \end{aligned}$$

where t is the royalty charge per m^3 mineral in concentrator feed in excess of the break-even quantity; then:

$$\frac{\delta\theta}{\delta g_c} = -tQ_m g_c f(g_c) + tQ_m g_r f(g_c) \text{ and}$$

$$\frac{\delta\theta}{\delta F} = \frac{\delta\theta}{\delta c_1} = 0.$$

After substitution into equations (17) to (19), and assuming, for simplicity, that the mining rate adapts perfectly to optimal concentrator feed rate, and that price-cost relations are steady, the compound cutoff grade decision is governed by:

$$g_c = \frac{c_0 - t g_r + \frac{f+rV}{F}}{y(s-c_1) - t}; \quad (20)$$

$$\frac{\delta y}{\delta F} = - \frac{f+rV}{(s-c_1)F^2 \bar{g}}; \quad (21)$$

$$\frac{\delta y}{\delta c_1} = \frac{y}{(s-c_1)}. \quad (22)$$

To judge the effect of this royalty on the cutoff grade, con-

sider the derivative of (20):

$$\frac{\delta g_c}{\delta t} = - \frac{g_r}{y(s-c_1)-t} + \frac{c_0 - t g_r + \frac{f+rV}{F}}{(y(s-c_1)-t)^2} \text{ or:}$$

$$\frac{\delta g_c}{\delta t} = - \frac{(g_r - g_c)}{y(s-c_1) - t}. \quad (23)$$

Since, obviously, the cutoff grade is below the break-even grade (as mining cost has been incurred whether material is dumped or milled), and, naturally, t will be less than the net value of mineral, both the first and second derivative of g_c with respect to t are always negative. The cutoff grade decreases with increasing rapidity as t increases. This form of royalty is therefore a powerful instrument in the hands of a government to increase mine recovery and mine life, without affecting the economic feasibility point. If t is expressed in terms of s , this tax would remain fair in fluctuating market situations, to which g_r is also linked. Concentrator recovery is a mixed aspect: on the one hand, the lower average grade will force an increasing feed rate and thus lower recovery by (21); on the other hand this lower recovery stimulates the use of more reagents to partially balance this effect, by (22).

As a second scheme, a form of resource rent tax based on operating profit is considered. A tax on return to equity is more generally advocated. However, if the resource rent tax cuts in after a threshold return on total equity has been earned, the government is disadvantaged, as its revenue is deferred; if the target rate of return to equity is converted to an equivalent annual rate of return, there is no essential difference from a tax on operating profit with a straight-line depreciation allowance in real terms, which will serve as an example.

In the calculation of operating profit artificial costs arise, for example, in the form of depreciation and other allowances. If these be represented per unit time by D , operating profit before tax is:

$$\begin{aligned} P &= Q_m \left\{ y(s-c_1) \int_{g_c}^{\infty} gf(g)dg - \right. \\ &\quad \left. - \left\{ c_0 + \frac{f+D}{F} \right\} \int_{g_c}^{\infty} f(g)dg - m \right\}. \end{aligned}$$

With ordinary taxation, tax would be extracted at the flat rate t on P . A government intent on collecting a greater share of resource rent might consider the imposition of a higher rate of taxation t_e , on operating profit before tax in excess of a certain fraction a of total allowable costs. These allowable costs sum up to:

$$\begin{aligned} C &= Q_m \left\{ y c_1 \int_{g_c}^{\infty} gf(g)dg + \right. \\ &\quad \left. + \left\{ c_0 + \frac{f+D}{F} \right\} \int_{g_c}^{\infty} f(g)dg + m \right\}. \end{aligned}$$

Let thus:

$$\Theta = taC + t_e(P-aC) = t_eP - aC(t_e-t).$$

With this scheme the effective tax burden would be less than the flat company tax rate if operating profit falls below the target aC , but this is consistent with the notion of economic rent. Flat company taxation is obtained as a limiting case, by setting $t_e=t$. The net contribution to operator's present value is:

$$v = (1-t_e)Q_m \left\{ ys \int_{g_c}^{\infty} gf(g)dg - \frac{1-t_e-a(t_e-t)}{1-t_e} \left\{ m + y c_1 \int_{g_c}^{\infty} gf(g)dg + \left\{ c_0 + \frac{f + \frac{rV-(t_e+a(t_e-t))D}{1-t_e-a(t_e-t)}}{F} \right\} \int_{g_c}^{\infty} f(g)dg \right. \right. \quad (24)$$

After the usual, now very tedious, manipulations this tax formula produces the following results:

$$g_c = (1-t_e-a(t_e-t)) \left\{ \frac{c_0 + \frac{f}{F} \frac{rV-(t_e+a(t_e-t))D}{F(1-t_e-a(t_e-t))}}{y(s(1-t_e)-c_1(1-t_e-a(t_e-t)))} \right\}; \quad (25)$$

$$\frac{\delta y}{\delta F} = -(1-t_e-a(t_e-t)) \left\{ \frac{f + \frac{rV-(t_e+a(t_e-t))D}{1-t_e-a(t_e-t)}}{(s(1-t_e)-c_1(1-t_e-a(t_e-t)))F^2g} \right\}; \quad (26)$$

$$\frac{\delta y}{\delta c_1} = \frac{y(1-t_e-a(t_e-t))}{s(1-t_e)-c_1(1-t_e-a(t_e-t))}. \quad (27)$$

Although these expressions appear rather unhelpful, inspection shows that for greater values of $a(t_e-t)$ the denominator of (25) increases and the numerator decreases. Simultaneously, by (26) $\delta y/\delta F$ would be less negative, and by (27) $\delta y/\delta c_1$ would also be less. Excess profits taxation thus results in better mine and better mill recoveries. It would seem that some care is required in setting the target profitability and the two tax rates t_e and t , if overall recovery is to be encouraged to the full. This would be a matter for negotiation between operator and government. If it were, for example, agreed that the effective tax rate should be 0.5 when gross profit is 16% of allowable cost, then with $t = 0.5$ and $t_e = 0.77$, the effective tax would be approximately 0.33 on a gross operating profit of 10% and 0.6 on a gross profit of 25%.

It may well be necessary to vary these values during the life of the project, to ensure that the present scheme will always lead to lower cutoff grades than a flat rate of taxation, as the

cutoff grade changes faster with mining progress under the present scheme. This may be seen as follows. If the reagent cost is, for convenience, assumed constant, it may be deducted from s and c_1 may be ignored in equation (25). Then:

$$\frac{\delta g_c}{\delta V} = \frac{r}{Fys(1-t_e)}. \quad (28)$$

In the case of flat rate taxation $t_e = t$, the ratio of the right-hand sides of equation (28) for flat-rate taxation to resource-rent taxation is therefore very nearly equal to $(1-t_e)/(1-t)$ (this is approximate, because F and y will not be quite the same under the two taxation schemes). The ratio of contributions to present value of one parcel is, however, much closer to unity than is $(1-t_e)/(1-t)$, because t_e is levied on excess profits only. These two observations imply that the ratio of $\delta g_c/\delta Q$ for flat taxation to that for resource rent taxation must be less than unity. Thus the cutoff grade will change more markedly with mining progress in the case of resource rent taxation.

In fact, the cutoff grade trajectories over time under the two schemes can cross. Using subscript 1 for flat-rate taxation and subscript 2 for resource-rent taxation, it follows from equation (25), with the above simplification, that the cutoff grade is raised by resource-rent taxation if:

$$(1 - a \frac{t_e-t}{1-t_e}) \left(\frac{c}{y_2s} + \frac{f}{F_2y_2s} \right) + \frac{rV_2-(t_e+a(t_e-t))D}{F_2y_2s(1-t_e)} > \\ > \frac{c}{y_1s} + \frac{f}{F_1y_1s} + \frac{rV_1-tD}{F_1y_1s(1-t)}.$$

Differences in concentrator recovery and feed rate can only be relatively small; if these are neglected the above condition becomes:

$$rV_2 - \frac{(1-t_e)rV_1}{(1-t)} > a(t_e-t)Fc + f + \frac{(1+a)(t_e-t)D}{(1-t)}. \quad (29)$$

Since the present value contribution v_1 at any stage is always less than $(1-t)v_2/(1-t_e)$, as t_e is levied on excess operating profits only, the difference on the left-hand side of this inequality keeps on increasing when moving backwards in time from the reference stage, whilst the right-hand side remains constant. The possibility of loss of extraction efficiency with the resource-rent tax scheme is thus a real one for high values of the remainder of the deposit, when a reduction in operating time T , by high-grading, lowers the opportunity cost of time rVT by a greater amount than it increases the tax on excess profits.

Both tax formulae considered shift some risk from the operator to the government.

CONCLUDING REMARKS

The impact liberation characteristics can exert on break-even grades has been demonstrated before (SCHAAP, 1979-b). In an earlier, less complete, analysis the effect of work index and effective mean mineral grain size upon cutoff grade has also been illustrated (SCHAAP, 1981). From this work it is clear that the cutoff grade decision needs to be a compound one, determining both mine and concentrator operating policies. The state-of-the-art in modelling and control of processing operations permits the cutoff grade theory advanced in this paper to be applied, given that ore quality parameters, other than grade, receive similar attention to that devoted to grade in the geostatistical assessments of deposits.

In addition to operational applications, the theory allows the impact of taxation upon mineral extraction to be studied directly, thus assisting in the design of a suitable tax formula, that will help the government to achieve its objectives. Recent agreements between international operators and host governments of developing countries, documented by MIKESSELL (1978), do, as yet, display little or no concern with specific characteristics of the deposits and their extraction efficiencies during the negotiations of the details of the taxation scheme under which these deposits are to be exploited. Yet, overall recovery may well be important particularly to developing countries. Greater recovery means an extended mine life, which allows more time for advancing the rest of the economy in preparation for the end of foreign exchange earnings from the mining project. This is, perhaps, the greatest opportunity mining can provide in such countries.

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