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THE ANGLE OF INITIAL YIELD OF HAPHAZARD ASSEMBLAGES OF EQUAL SPHERES IN BULK

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ABSTRACT

The maximum slope angle assumed by heaped granular solids is of considerable interest in geology, geomorphology and soil mechanics, as also is the slope stability of such heaps. Representing the solids by equal spheres, we here deduce that the relationship between the maximum slope angle (angle of initial yield) and the concentration of equal spheres arranged haphazardly in bulk in the gravity field is

$$\tan \phi_i = k_1 C - k_2 + A,$$

in which ϕ_i is the angle of initial yield, C is the fractional volume concentration, k_1 and k_2 are known constants depending on the properties of equal spheres in regular cubical and rhombohedral array, and A is a variable dimensionless coefficient representing frictional, electrostatic and other non-gravitational forces.

This relationship is broadly confirmed by experiments using glass beads of two different narrow size ranges, though the experiments are not of a high order and accuracy. It is of interest that a natural sand, well sorted compared with other sands but still showing an approximately 1 : 4 range of sizes, also conforms to the relationship deduced for equal spheres.

INTRODUCTION

It is well known that a plane surface underlain by loose granular solids cannot be built up in the gravity field above a certain critical slope angle without an avalanche of grains being induced to flow down the slope. It is also well known that the angle of the surface relative to the horizontal after the avalanche has ceased to flow is a little smaller than the angle at the start of the avalanching. The larger of this pair of angles is the *angle of initial yield*, designated by ϕ_i, while the smaller is the *residual angle after shearing*, denoted by ϕ_r, and commonly reported as the angle of repose (V a n B u r k a l o w, 1945; C a r r i g y, 1967). Then

$$\phi_i = \phi_r + \Delta \phi, \tag{1}$$

in which

$$\Delta \phi = \phi_i - \phi_r.$$

Although the angle of initial yield has been deduced theoretically for *single* particles on a rough bed (E a g l e s o n and D e a n, 1959; M i l l e r and B y r n e, 1966), there has hitherto been no theory for the equally if not more significant angle of initial yield of granular solids arranged haphazardly in bulk. In this paper we shall advance and test a theory for ϕ_i with particular reference to bulk assemblages and to the quantity $\Delta \phi$.

Although the quantity $\Delta \phi$ is less fundamental than ϕ_i, it is of interest in geology, geomorphology and soil mechanics in connection with the physical character and behavior of inclined surfaces, such as the slip faces of ripples or dunes and man-made or natural screes, that are fed with fresh granular material.

For it can be shown that the period of avalanches on such a slope is given by

$$T = T_{\infty} + T_o \left(\frac{T}{T_{\infty}}\right)^n, \tag{2}$$

in which T is the avalanche period in seconds and n is an empirical exponent ($1>n>-5$) depending on the

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material and the conditions of avalanching. The other quantities, measured in seconds, are defined as

$$T_o = \frac{L}{U},$$

and

$$T_\infty = \frac{\Delta\phi}{\left|\frac{dW}{dx}\right| \frac{1}{\rho} \cos \phi_r},$$

wherein L is the length in centimetres of the slope down which the avalanches travel, U is a velocity in cm/sec characteristic of the avalanches, ρ is the bulk density in gm/cm^3 of the granular solids fed to the slope, dW/dx is the gradient of the weight feed rate W measured in gm/cm^2/sec of solids to the slope, and $\Delta\phi$ is measured in radians/sec. It is taken that x is measured horizontally and that $x=0$ corresponds to the upper limit of the extent of the slope; on a dune slip face or a scree slope dW/dx would be measured as a negative quantity. In the practical case we have $T_\infty \gg T_o$, so that from eq. (2) $T \approx T_\infty$ and

$$T \sim \Delta\phi \tag{3}$$

for a constant feed rate gradient and bulk density.

Secondly, it follows from considerations outlined by Bagnold (1954, 1967) that surfaces underlain by granular solids can collapse to form free-running avalanches if subjected to large enough downslope accelerations. If ϕ is the initial angle of such a slope, and $\phi_i > \phi > \phi_r$, then

$$a = a_{max} \frac{(\phi_i - \phi)}{\Delta\phi}, \tag{4}$$

in which

$$a_{max} = g(\tan \phi_i - \tan \phi_r),$$

wherein g is the acceleration due to gravity. In eq. (4) a is the downslope acceleration which will just cause an avalanche on a slope of initial angle ϕ and a_{max} is the maximum possible downslope acceleration that will initiate a free-running avalanche. It will be seen from eq. (4) that

$$a \sim \Delta\phi^{-1} \tag{5}$$

for constant values of the other quantities involved.

The magnitude of $\Delta\phi$ is therefore an index to the stability characteristics of a slope underlain by granular solids, but it can be found only if ϕ_i and ϕ_r are known. Eq. (3) can be interpreted to mean that $\Delta\phi$ governs the probability of an avalanche occurring on a slope as a consequence of the continued feeding of the slope. Eqs. (4) and (5) are criteria for the response of the slope to externally applied disturbing forces arising from, for example, impacts or shock waves.

It will be recalled from eq. (1) that $\Delta\phi$ is the difference between two other quantities. Experiments I have made in connection with eq. (2), which will be reported elsewhere, showed that while ϕ_r is very nearly constant for a given material regardless of the conditions of avalanching, ϕ_i is subject to large variations. Carrigy (1967) also found little variation in ϕ_r for a wide range of materials and conditions of avalanching. Thus $\Delta\phi$ is not a constant. It approaches 8° for naturally occurring quartz sands, and can be as large as 14° for sands composed of very irregular shaped particles. The experiments referred to pointed to $\Delta\phi$ being a direct function of the volume concentration of the granular solids, an inference which we shall justify below. We may therefore write

$$\tan \Delta\phi = f_1 (C), \tag{6}$$

or, since $\Delta\phi$ is the difference between two more fundamental quantities,

$$\tan \phi_i = f_2 (C), \tag{7}$$

in which C is the fractional volume concentration defined as

$$C = \frac{\text{Space occupied by solids}}{\text{Total space}}$$

It was shown by Kolbuszewskii (1948a, 1948b, 1950a, 1953) that C depended on the conditions of deposition of the solids. MacRae and Gray (1961) found the elasticity of the particles strongly influenced C. It is now possible to present a simple theory representing a solution to eq. (7) and appropriate to granular solids in bulk.

THEORY

Consider first the equilibrium in a vertical plane in the gravity field of three equal rigid frictionless spheres of radius R whose centres lie in the plane specified (fig. 1). Two spheres are in contact and their centres lie on a common horizontal. The third sphere

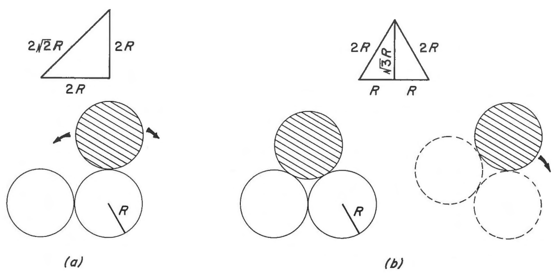

Fig. 1
Equilibrium in a vertical plane in the gravity field of a single rigid frictionless free sphere on two fixed ones.

is free to take up any position above the other two, provided always that it remains in contact with at least one of the fixed spheres. When the centre of the free sphere lies vertically above the centre of one of the fixed spheres, the free sphere is in unstable equilibrium and is free to roll to right or left without any change of attitude of the fixed spheres being required (fig. 1a). For such an arrangement the angle of yield is zero. However, when the free sphere rests in the hopper formed by the other two (fig. 1b), the line of centres of the fixed spheres must be tilted to right or left by an angle of yield of tangent $1/\sqrt{3}$ before the free sphere will roll away. This system is stable for all angles of tilt whose tangent is less than $1/\sqrt{3}$.

The difference between these two cases can be considered one of looseness of packing, or volume concentration. In the isoscles arrangement (fig. 1a), the mean distance between sphere centres is

$$d = \frac{R(2+2+2\sqrt{2})}{3} = 2.276R,$$

whereas for the equilateral arrangement of centres (fig. 1b), we find

$$d = \frac{R(2+2+2)}{3} = 2R.$$

Now d is related to the 'linear' concentration (B a g n o l d, 1954) of the solids through

$$\lambda = \frac{2R}{(d-2R)},$$

in which λ is the linear concentration. We also have that

$$\lambda = \frac{1}{(C_*/C)^{\frac{1}{3}} - 1}, \qquad (8)$$

wherein C is the observed fractional volume concentration and C_* is the maximum possible concentration. For equal spheres packed in a regular manner, C is a maximum when the spheres are in rhombohedral array (G r a t o n and F r a s e r, 1935), whence $C_* = \pi/3\sqrt{2} = 0.741$.The minimum value of C for a regular packing is found when the spheres are in cubical array (G r a t o n and F r a s e r, 1935), for which $C = \pi/3\sqrt{4} = 0.524$. There are, as discussed by G r a t o n and F r a s e r (1935), two other possible regular arrangements of equal spheres. In the orthorhombic arrangement, $C = 0.605$, while for the tetragonal mode of packing, $C = 0.698$. Thus for regular packings the fractional volume concentration of solids is not a continuous variable.

But in the practical case, whether of equal spheres or of irregular shaped grains of mixed sizes, the concentration is a continuous variable, within certain limits, and the packing of the solids is haphazard and not regular. As the difficulties of the statistical geometry of haphazard sphere packings have not yet been conquered, in spite of substantial advances by B e r n a l (1959), 1964), our problem is to develop the model for the relationship between yield angle and C exemplified by figure 1 in terms of the properties of regular arrangements of spheres.

Following S m i t h, F o o t e and B u s a n g (1929), we shall assume that the bulk properties of haphazard assemblages of spheres can be statistically represented by a model in which a fraction x of spheres in rhombohedral packing is combined with a fraction $(1-x)$ in cubical array to yield the observed concentration of the haphazard packing. Then

$$C = C_{rbc}x + C_{cbc}(1-x), \qquad (9)$$

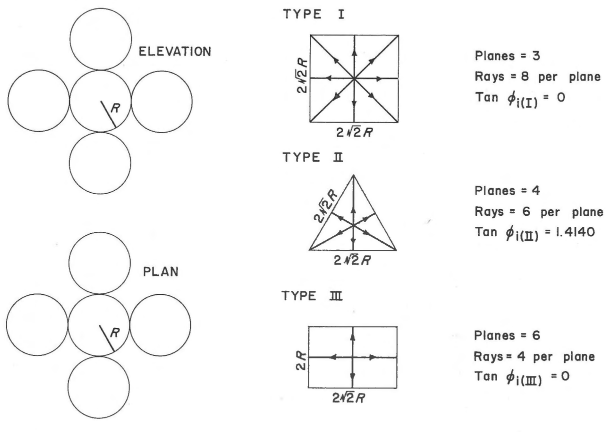

Fig. 2
The unit cubical array of equal spheres and its stability planes and rays.

and

$$x = \frac{C \ - C_{cbc}}{C_{rbc} - C_{cbc}}, \qquad (10)$$

in which $C_{cbc} = 0.524$ and $C_{rbc} = 0.741$. We therefore propose as a solution to eq. (7)

$$\tan \phi_i = \tan \phi_{i(rbc)} x + \tan \phi_{i(cbc)} (1 - x) + A, \quad (11)$$

in which $\phi_{i(rbc)}$ and $\phi_{i(cbc)}$ are the angles of initial yield of equal spheres in rhombohedral and cubical array respectively, and the spheres in both cases are assumed rigid and frictionless. The dimensionless term A is added to include the possible effects in real cases of different types of friction between particles, electrostatic attractions, and electrochemical forces, all of which can modify the theoretical angle of initial yield. That is

$$A = A_1 + A_2 + A_3 \text{ etc.},$$

where A_1, A_2, and A_3 are dimensionless coefficients appropriate to the different sources of influence. It will be seen that eq. (11) is linear, with a slope of (tan $\phi_{i(rbc)}$ − tan $\phi_{i(cbc)}$) and an intercept of (tan $\phi_{i(cbc)}$ + A). The coefficient A is variable from one material and situation to another and is unlikely to be so small that it can be neglected. Some indication of the contribution of, for example, sliding friction to A can be got from an experimental result of C a r r i g y (1967), who found that ϕ_i for polished steel ball bearing was 25° approximately but for the same ball bearings in a slightly rusted condition was 34° approximately.

The remaining problem is to find numerical values for tan $\phi_{i(rbc)}$ and tan $\phi_{i(cbc)}$. To do this we shall consider, with respect to unit arrays of rhombohedral and cubically arranged equal spheres, the stability of a single free sphere either resting upon a single fixed sphere or lying in the hopper formed by an arrangement of fixed spheres, either three or four in number, whose centres lie in a common plane. In any arrangement, the angle of yield is reached when the centre of the free sphere lies vertically above either the centre of a fixed sphere or the line joining the centres of two adjacent fixed spheres.When the free sphere rests on a fixed sphere, all paths of rolling are the same. When the free sphere lies in a hopper, however, it may roll away either directly over the centre of a fixed sphere or through the gap between two adjacent fixed spheres. In this case there are two different alternative angles of yield, depending on the immediate path of rolling. The properties of certain extended regular arrays of spheres justify in these cases the use of the larger yield angle as characteristic of the arrangement.

It will be found that the centres of the fixed spheres define in the common plane of centres regular geometrical figures, either squares, rectangles or equilateral triangles. In the case of a free sphere emerging from a hopper, these figures are found to be bisected by certain paths of rolling which for convenience will be called rays. Analogous rays can be drawn for a free sphere in contact with a single fixed sphere. It will be assumed, subject to some further considerations, that the required values of tan $\phi_{i(rbc)}$ and tan $\phi_{i(cbc)}$ are the weighted means of the tangents for the different arrangements of free and fixed spheres in the appropriate unit array. This approach is a little arbitrary, but its use can be justified in connection with an elementary theory, such as the present one, by pointing to the fact that the total rays for each regular packing have a substantially random (i.e. uniform) arrangement in space. In applying the values of the tangents so obtained, we assume in our model after F o o t e, S m i t h and B u s a n g (1929) that the different rhombohedral and cubical units are randomly oriented in space.

Figure 2 summarises the relevant properties of the unit cubical array of equal spheres. The planes of type I, three in number, are squares of side $2\sqrt{2}R$. Each plane contains the centres of five spheres, one at each corner and one at the centre, and therefore passes through the geometrical centre of the unit array. There are eight rays in each plane and the value of tan $\phi_{i(I)}$ for the plane is zero. The planes of type II are equilateral triangles of side $2\sqrt{2}R$ with a sphere

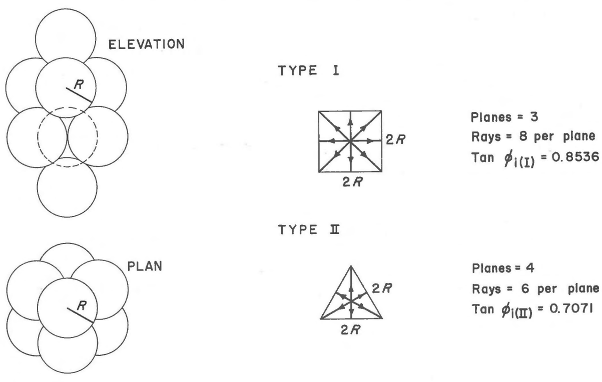

Fig. 3
The unit rhombohedral array of equal spheres and its stability planes and rays.

at each corner. There are four such planes, each with six rays, and tan $\phi_{i(II)}$ is 1.4140. Notionally, when several unit cubical arrays are arranged together, there is a third type of plane which is a rectangle of sides $2R$ and $2\sqrt{2}R$ with a sphere at each corner. Each plane of this type has four bisecting rays. The value of tan $\phi_{i(III)}$ is zero for all rays parallel to the shorter sides, but is unity for the rays parallel to the longer ones. However, since the rays parallel to the sides of length $2\sqrt{2}R$ are coincident with rays in planes of type I for which tan $\phi_{i(I)}$ is zero, we shall take it that tan $\phi_{i(III)}$ is also zero. Then, by our previous definition,

$$\tan \phi_{i(cbc)} = \frac{(24 \times 0) + (24 \times 1.4140) + (24 \times 0)}{72}$$

$$= 0.4713, (\phi_{i(cbc)} = 25°14').$$

This value is interpreted as an upper bound on tan $\phi_{i(cbc)}$ rather than as a definitive value. For inspection of figure 2 shows that the planes of type I, for which tan $\phi_{i(I)}$ is zero, are orthogonally related and pass through the central sphere of the unit array, dividing the array into eight equal segments. Clearly the array has a negligible inherent stability, and tan $\phi_{i(cbc)}$ could well be zero on account of the symmetry and pervading influence of the type–I planes.

The relevant properties of the unit rhombohedral array are shown in figure 3. The three planes of type I are squares of side $2R$ with a sphere at each corner. Each plane contains four rays parallel to the sides for which the tangent of the angle of yield is 0.7071. The tangent of the yield angle for the diagonal rays is unity. Because this is a case of a free sphere in the hopper formed by *four* other spheres, we shall take it that tan $\phi_{i(I)} = 0.8536$, the mean of the value for the parallel and diagonal rays. The planes of type II are equilateral triangles of side $2R$ with a sphere at each corner. Each plane contains six rays for which the maximum tan $\phi_{i(II)} = 0.7071$. We therefore obtain

$$\tan \phi_{i(rbc)} = \frac{(24 \times 0.8536) + (24 \times 0.7071)}{48}$$

$$= 0.7803, (\phi_{i(rbc)} = 37°56').$$

Clearly the unit rhombohedral array is a tight arrangement with a high inherent stability. There are no reasons for regarding the value of the angle of initial yield obtained as an upper bound.

In view of the ambiguous value taken by tan $\phi_{i(cbc)}$, we propose the following forms of eq. (11) as upper and lower bounds on the relationship between tan ϕ_i and C

$$\tan \phi_i = 0.7803x + A, \tag{12}$$

and

$$\tan \phi_i = 0.7803x + 0.4713(1-x) + A. \tag{13}$$

Introducing eq. (9) and evaluating the numerical constants, we can write eqs. (12) and (13) as

$$\tan \phi_i = 3.596\, C - 1.884 + A \tag{14}$$

and

$$\tan \phi_i = 1.424\, C - 0.275 + A. \tag{15}$$

It will be seen that eq. (14) has more than twice the slope of eq. (13). Eqs. (12) and (13) give tan ϕ_i in terms of x, whereas eqs. (14) and (15) relate tan ϕ_i directly to the concentration.

In the practical case of haphazardly arranged equal spheres, C cannot extend over the full theoretical range $0.741 \geqslant C \geqslant 0.524$. Experiment shows that there is a maximum and a minimum possible value of C for a haphazard assemblage. According to Scott (1960), who applied to his experiments the necessary corrections, the maximum possible concentration of haphazardly packed equal spheres is $C_{max} = 0.637$, while the minimum possible concentration is

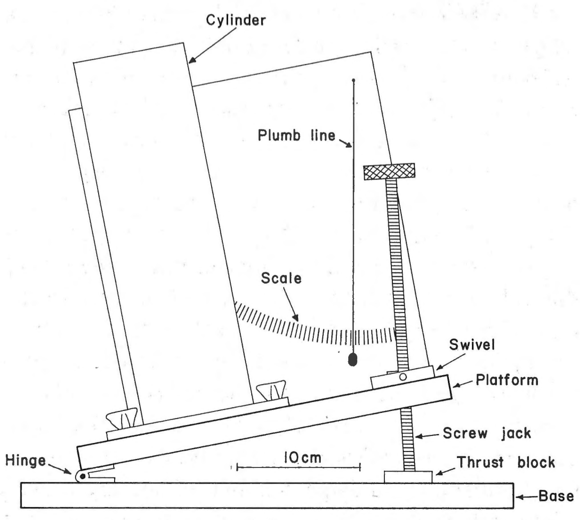

Fig. 4
Apparatus used to determine the tangent of the angle of initial yield as a function of the fractional volume concentration of granular solids.

$C_{min} = 0.601$. A similar narrow but rather lower range of possible values is found in the case of haphazard assemblages of granular solids of mixed particle sizes, for example, naturally occurring sands. In the case of equal spheres, with friction and other external influences neglected, we therefore find an upper possible limit to ϕ_i in the range $32^\circ\ 18 \geqslant \phi_i \geqslant 22^\circ\ 7'$, and a lower possible limit in the range $30^\circ\ 7' \geqslant \phi_i \geqslant 15^\circ\ 30'$. Naturally eq. (14) provides the largest difference between the upper and lower limits on ϕ_i. The largest values of ϕ_i within the possible range are given by eq. (15).

EXPERIMENT

The object of the present experiments was to test eq. (11) by measuring ϕ_i as a function of C calculated from measured volumes and weights. It will be obvious from the preceding discussion of the expected ranges of ϕ_i and C in practical cases that both quantities must be obtained to a high order of accuracy.

The experiments were carried out in the simple apparatus shown in figure 4. This consists of a hard-wood base, firmly attached to an acceptably rigid laboratory bench, which bears a hard-wood platform hinged at one end that can be tilted by means of a smoothly running screw jack. The platform carries a large protractor scale with a small plumb-line hinged at the origin. Angles could be read from this scale to $\pm 0.05^\circ$ arc, giving an error in tan ϕ_i of approximately ± 0.001 over the range of interest. Attached to the tilting platform by means of wing nuts is a removeable 'Perspex' cylinder, open at the top, of 8.3 cm internal diameter and capacity about 1800 ml. The long axis of the cylinder is accurately perpendicular to the tilting platform and the rim of the cylinder is smooth and accurately square to the long axis. Using a point gauge carried on the rim of the cylinder, the level of the top of a mass of glass beads or sand poured into the cylinder could be measured at any desired point to ± 0.01 cm. In practice the mean level was determined from gauge measurements at a large number of closely spaced points arranged on a square grid. The cylinder was calibrated using water so that the mean of the point gauge readings could be converted directly into volume. Knowing the weight of material in the cylinder, and its bulk volume and measured specific gravity, the mean fractional volume concentration could be calculated to ± 0.0005, giving a measurement error in x of approximately ± 0.002.

It should be carefully noted that in these experiments it is assumed that the concentration measured for the total mass in the cylinder is the concentration relevant to that part of the granular material which forms an avalanche when the cylinder is tilted to ϕ_i. It is probable that in this assumption lies a variable error much larger than any of the errors of measurement. Moreover, in the discussion of eq. (4) it was seen that as $\phi \rightarrow \phi i$, a slope becomes increasingly sensitive to small disturbing accelerations. Disturbances could not entirely be avoided in the present experiments, so that the measured values of ϕ_i could fall to varying extents below the true values. This effect is likely to be greatest at the smaller concentrations, when $\Delta\phi$ is relatively small.

The materials used in the experiments were two sizes of glass beads, obtained from the Cataphote Corporation of U.S.A., and a quartz beach sand collected on the Lancashire coast through the kindness of Mr. R.G.C. Bathurst of Liverpool University. The larger beads had a size range between 0.590 and 0.840 mm diameter while the finer ones ranged between 0.210 and 0.297 mm diameter. For both samples of beads the border diameter ratio is 1 : 1.42.

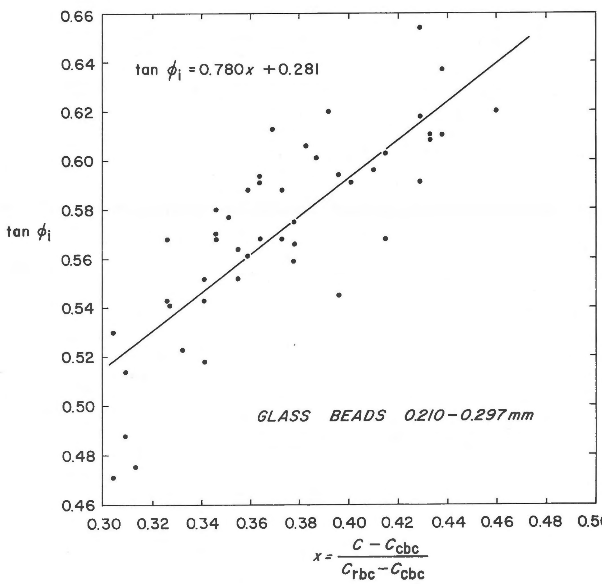

Fig. 5
Experimental relationship between tan ϕ_i and C for glass beads of size range 0.210-0.297 mm diameter.

These beads should therefore behave little differently from equal spheres. The beach sand had the following mechanical analysis.

Grade	Wt.%
0.25–0.35 mm	5.9
0.18–0.25	63.4
0.13–0.18	30.3
0.088–0.13	0.4
	100.0

Although very well sorted as natural sands go, this sand has a border diameter ratio of 1 : 3.98. It was intuitively expected to behave rather differently from the glass beads.

A typical experiment with the apparatus proceeds as follows. The granular material is poured into the cylinder, detached from the tilting platform, to within about 5 cm of the top. The top of the cylinder is now closed off by pressing against the rim a plate faced with a thick layer of soft rubber. Scott's (1960) procedure is then used to obtain the minimum concentration of the material. The cylinder is tipped horizontally and then slowly turned about its axis and gradually returned to the vertical position, whereupon it is carefully fixed on the tilting platform. After some practice at this procedure, a flat surface could be obtained on the material contained in the cylinder. The flat surface was not necessarily horizontal when the cylinder was vertical, but in view of the method adopted in measuring level with the point gauge, this was not objectionable as a correction for surface tilt was easily made to good accuracy. If a concentration larger than that associated with loosest haphazard packing was required, the bench on which the apparatus was mounted was tapped with a hammer delivering controlled blows. Prolonged tapping increased the concentration to that appropriate to closest haphazard packing. It was assumed that the increase of concentration following tapping occurred uniformly thoughout the mass in the cylinder. After the surface of the material had been measured with the gauge, the cylinder was slowly and smoothly tilted by means of the screw jack until an avalanche occurred. The protractor scale was then read and the angle recorded. Between 40 and 50 sets of observations were made in connection with each of the three experimental materials.

Since in the experiments, as in nature, the avalanching occurred under three-dimensional conditions, it was anticipated that the largest contribution to the dimensionless coefficient A in eq. (11) would come from sliding friction. The coefficient of sliding friction was found for each of the experimental materials using the tilting platform shown in figure 4. The beads were mounted with a thin film of canada balsam on to a glass slip, the whole being allowed to cool on a flat surface with the beads facing downward. This slip, with the beads downward, was placed on a sheet of polished glass mounted flat on the tilting platform. The platform was then gradually tilted until the slip began to slide down the resulting inclined plane. The reading on the protractor scale gave, when corrected for systematic error, the angle of friction. In the case of the beach sand, a slip prepared as for the beads was allowed to slide down a sheet of glass whose surface had been finely ground. It is believed that this type of surface approximates closely in texture to the surface of natural quartz

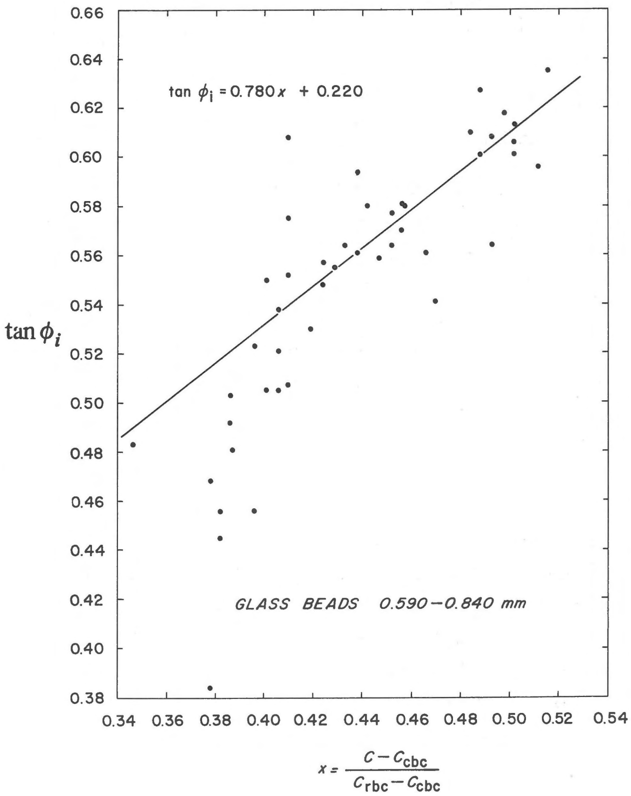

Fig. 6
Experimental relationship between tan ϕ_i and C for glass beads of size range 0.590-0.840 mm diameter.

grains. Ten observations were made on each material and the coefficient of sliding friction was obtained by averaging the tangents of the observed friction angles. The results are as follows.

Material	Mean coefficient
Finer beads	0.158
Coarser beads	0.175
Beach sand	0.439

The coefficients for the two sizes of beads are much the same, but that for the beach sand is nearly three times as large.

Figure 5 shows the experimental relationship between $\tan \phi_i$ and the calculated quantity x as obtained for the finer glass beads. The range of x, between 0.304 and 0.460, is a rather lower than the range expected for haphazard packings of equal spheres ($0.355 \leqslant x \leqslant 0.521$) but overlaps with this range. Although there is substantial scatter in the data, due no doubt to the probably large and uncontrolled sources of error already discussed, the equation.

$$\tan \phi_i = 0.780x + 0.281 \tag{16}$$

provides a very fair approximation to the results. The data fall furthest below the curve for small values of x, which was expected.

The results obtained with the coarser beads are shown in figure 6. There is a similar scatter in the data as in figure 5, but the range of values of x, between 0.346 and 0.512, is now almost the same as the expected range. For the coarser beads the equation

$$\tan \phi_i = 0.780x + 0.220 \tag{17}$$

provides an acceptable approximation to the data. Again the observed values of $\tan \phi_i$ fall farthest below the curve for the smaller experimental values of x.

These experiments are not of a high order of accuracy, but it is clear from the results given in figures 5 and 6 that eq. (14) and not eq. (15), is the relevant relationship between $\tan \phi_i$ and C. This interpretation of the results implies that the true value of $\tan \phi_{i(cbc)}$, which emerged ambiguously from the analysis, is $\tan \phi_{i(cbc)} = 0$. Although the slope of the experimental curves appears indistinguishable from the theoretical gradient, the experimental values of $\tan \phi_i$ are substantially larger than the theoretical ones. Since equality of gradient is assumed, the difference between theoretical and experimental values is given by the magnitude of the coefficient A. Putting A_f as the measured coefficient of sliding friction, and A_x as a coefficient representing influences still unindentified, we have for the fine beads

$$A = 0.281, \quad A_f = 0.158, \quad A_x = 0.123,$$

and for the coarser ones

$$A = 0.220, \quad A_f = 0.175, \quad A_x = 0.045.$$

The smaller value of A_x for the larger beads is consistent with the observed smaller electrostatic effects with this material, although the lower range of values of x appears to account for a part of the difference.

It was found for the beach sand that $\tan \phi_i$ increased with ascending concentration (figure 7). The data scatter much as before but the equation

$$\tan \phi_i = 3.596C - 1.884 + 0.617 \tag{18}$$

is not out of keeping with the results obtained. It has the same gradient as eq. (14) expressed in terms of concentration. In eq. (18) $A = 0.617$, but A_f is measured as 0.439, so $A_x = 0.178$. It will be noticed from figure 7 that the range of concentrations obtained for the beach sand is broader and lower than that expected for equal spheres in haphazard packing. Although the beach sand seems to behave in a similar way to that expected of equal spheres, it is not

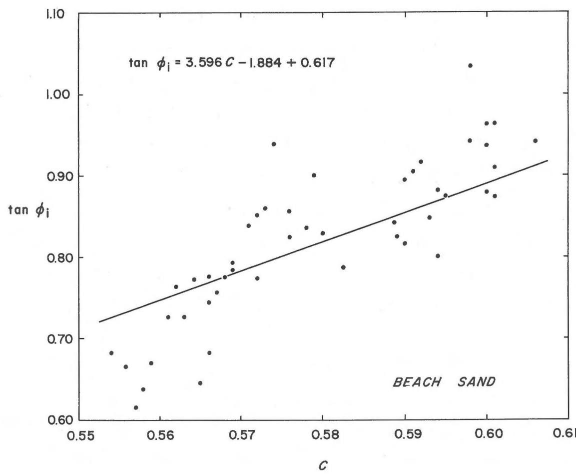

Fig. 7
Experimental relationship between $\tan \phi_i$ and C for a naturally occurring beach sand of very fine sand grade.

obvious that in such a case C_{rbc} and C_{cbc} can have any physical significance. However, it is equally not obvious that regular arrangements can not be achieved, or at least approached, in the case of mixtures of non-spherical particles of arbitrary size distribution. The difficulties of proving this are, however, formidable and the problem if hitherto recognised does not appear to have been solved.

SUMMARY OF CONCLUSIONS

The problem of the initial angle of yield of granular solids haphazardly arranged in bulk in the gravity field is one of extreme difficulty that concerns the statistical geometry of frictional solids heaped together through the operation of sedimentation processes of variable character. It has not yet been solved completely, but it is relevant to the stability of many natural and man-made slopes of interest in geology, geomorphology and soil mechanics.

The present work shows, however, that the yield angle of a haphazard assemblage of equal spheres can be obtained if it supposed that the assemblage is statistically equivalent to an arrangement in which separate units of rhombohedrally or cubically arrayed spheres are combined as fractions to yield the observed concentration of solids. In this connection it is shown that the tangent of initial yield of a frictionless unit cubical array is zero. In the case of a rhombohedral array, the tangent of the angle of yield is deduced to be 0.7803 for frictionless conditions. The tangent of the yield angle of a haphazard bulk assemblage is the sum of the products of the appropriate unit-array fractions and tangents of yield angle.

Experiments made with narrowly graded glass beads, though not of a high order of accuracy, give satisfactory support to the simple theoretical analysis. However, the observed values of the tangent of the angle of initial yield are larger than the theoretically expected ones, it appears chiefly because of sliding friction between the particles. It was also found that a natural beach sand behaved in the manner predicted theoretically for equal spheres. Frictional effects appeared again to increase the observed values above the theoretical ones.

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