THE SYSTEMATIC PACKING OF PROLATE SPHEROIDS WITH REFERENCE TO CONCENTRATION AND DILATANCY

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ABSTRACT

The use of the sphere as the ideal sedimentary particle is criticised and the prolate spheroid (ellipsoid of revolution) is proposed as a more realistic alternative.

Equal prolate spheroids identically oriented in space can be packed in six ways analogous to the six packings of equal spheres. The volume concentration of spheroids in systematic packing is identical with the concentration of spheres in the equivalent packing, except in cases of "cubic" packing in which concentration is a function of spheroid orientation and axial ratio. The dilatation angle of assemblages of prolate spheroids is a function of type and orientation of packing and of spheroid orientation relative to the direction of displacement. The maximum angle of initial yield of packings of spheroids is also dependent on type and orientation of packing and on spheroid orientation. The implications of these findings for the steepness and stability of slopes formed on loose granular materials are discussed.

INTRODUCTION

The properties and behaviour of granular solids in bulk are important in many branches of science and technology. In geology, for example, they figure in the problem of the amount of fluid that can be contained in a sand or gravel reservoir deposit and in the related question of the ease with which the fluid can be extracted from its host. Also, they bear on several problems of sedimentation: in the analysis of particle equilibrium on a plane grain-bed, in the behaviour of grains during fluid- or gravity-induced shearing, and in the stability of slopes formed on incoherent, granular deposits. The last case is well illustrated by the instability, with consequent avalanching, of rock screes and the lee slopes of ripples and dunes.

It has generally been assumed in the theoretical treatment of the properties and behaviour of granular solids, that the real particles of interest can ideally be represented by equal spheres. There exist several more or less exhaustive treatments based on this assumption, though these are not all relevant to every issue (Bagnold, 1954, 1967; Bernal, 1964, 1965; Deresiewicz, 1958; Graton and Fraser, 1935; Orr, 1966). The assumption of the sphere is undoubtedly valid for the investigation of many problems that involve real or imagined particles. It is, moreover, an attractive one in that the sphere has a peculiarly simple geometry, and is available cheaply and in many different sizes and materials for experimental purposes.

Unfortunately, the assumption of the sphere as the ideal sedimentary particle is inadequate for many geological purposes. Whilst our experience teaches that sedimentary particles evolve in form towards the sphere as they undergo wear and solution, it is also clear that real particles are practically never spherical and often radically unlike spheres. Instead, real particles approximate to ellipsoids, as demonstrated by the meticulous work of Sneed and Folk (1958) and of Moss (1962, 1963, 1966). The ellipsoid representing the typical real particle has a long dimension about 1.5 times the intermediate dimension and about 2 times the short dimension. Since form has been shown to have a profound effect on particle behavior in simple as well as complex

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sedimentary environments (Krumbein, 1942; Albertson, 1953; Flemming, 1964, 1965; Bluck, 1967), there is a strong case for a theoretical treatment of granular solids that is not based on the sphere as the ideal particle. A further justification is the observation that most natural detrital sediments of sand or gravel size possess an anisotropic particle fabric to which is broadly related an anisotropic permeability property (Potter and Mast, 1963; Mast and Potter, 1963). Assemblages of spheres are anisotropic as regards permeability only under special conditions of systematic packing. All non-random assemblages of ellipsoids are, however, anisotropic in terms of permeability.

The purpose of this paper is therefore to outline a theoretical treatment of some of the important properties of granular solids in bulk using the ellipsoid for the form of the ideal particle. For simplicity, the ellipsoid chosen is the prolate spheroid generated by rotating a plane ellipse about its major axis. Particular attention will be directed in the present paper towards the volume concentration and dilatation of systematic packings of such ellipsoids, the bodies being assumed of equal size.

GENERAL

The geometry of the plane ellipse (fig. 1) is discussed by Coxeter (1961) and Lockwood (1961). On the major axis AA′ are two foci, S and S′, at an equal distance from the geometrical centre C.

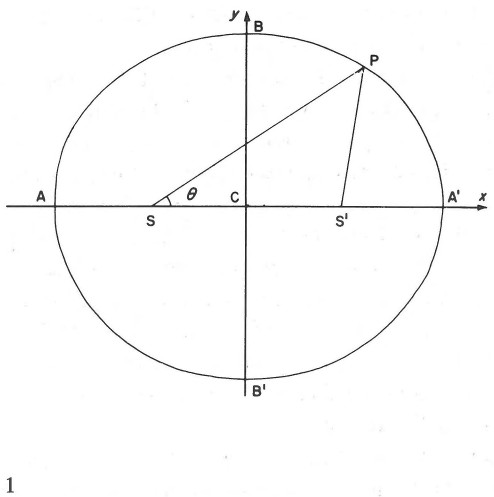

Fig. 1
Geometrical features of the plane ellipse.

We denote the major semi-axis CA by a and the minor semi-axis CB by b. The ratio CS : CA is the eccentricity e of the ellipse, which varies between 0 and 1 and is a measure of the flatness of the figure, as also is the axial ratio b/a. The length SP is denoted by r, the angle $\angle$ A′SP by θ. With these conventions, the position of any point P on the ellipse is given either by the polar equation (origin at focus)

$$r = \frac{a(1-e^2)}{(1-e\cos\theta)} \quad , \tag{1}$$

or by the Cartesian equation (origin at centre)

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \tag{2}$$

or by the parametric equations (origin at centre)

$$x = a\cos t\,,\quad y = b\sin t. \tag{3}$$

We also have that the volume of a prolate spheroid is $4/3\,\pi ab^2$.

PACKING AND VOLUME CONCENTRATION

It will be apparent that a prolate spheroid, unlike a sphere, can be assigned an orientation in space and that this orientation has an important bearing on the geometrical properties of systematic assemblages of such spheroids.

In this paper the orientation of a prolate spheroid in an assemblage of equal spheroids, as well as the orientation of the assemblages themselves, will be described by reference to mutually orthogonal coordinate axes (x, y, z). The different modes of simple, systematic packing of bodies such as equal spheres or equal prolate spheroids can all be illustrated by arranging eight of the bodies so as to be in contact in two layers each of four bodies, the centres of the bodies in each layer lying in a plane parallel to (x, z). It will also be understood that two of the rows of bodies in each layer are parallel to the x-axis. In almost all the simple, systematic arrangements to be discussed, the major axes of the eight spheroids will be assigned the same orientation in space. Two limiting orientations are of interest in each type of packing, when the major axes lie parallel to the y-axis (denoted by $a \parallel y$), and when the major axes lie parallel to the x-axis (denoted by $a \parallel x$). In the first case the spheroids lie on end, and in the second rest in their most stable position relative to the (x, z) plane. Intermediate orientations are possible in which the major axes lie in planes parallel to (x, y) but are inclined at an angle ψ to the x-axis. There is one possible systematic packing of equal prolate spheroids in which the major axes of four spheroids are parallel to the x-axis while the major axes of the remainder lie parallel to the y-axis, in a "card-house" arangement.

Four equal tangent spheres can be so arranged in a layer that their centres lie at the corners of either a square or a rhomb. As Graton and Fraser (1935) have demonstrated, there are three different ways of systematically stacking two square layers of spheres, one over the other, and three ways of systematically arranging two rhombic layers, making six different ways in all. However, two pairs from amongst these six represent the same fundamental type of packing but viewed in a different orientation. There are therefore only four fundamental, simple and systematic ways of packing equal spheres. In space-lattice terminology, these packings are the cubic, orthorhombic, tetragonal-sphenoidal, and rhombohedral.

Equal prolate spheroids whose major axes are parallel can be simply and systematically packed in six different ways analogous to the six different arrangements of equal spheres. But the shapes formed by joining together the centres of the spheroids are not identical with the shapes formed by the centres of spheres in the arrangement with which analogy is drawn. Thus a cube is obtained by joining together the centres of eight spheres divided between rows mutually at right angles. The geometrical shape formed by joining together the centres of eight equal and identically oriented prolate spheroids divided between rows mutually at right angles is a rectangular parallelepiped, of which two faces are squares of side $2b$ and four are rectangles measuring $2b$ by $2a$. Although there is no rigorous identity between simple systematic packings of spheres and prolate spheroids, it will be convenient for purposes of naming and comparison to apply to the different packings of spheroids the terms proposed by Graton and Fraser (1935) for spheres.

Figs. 2 and 3 show the six principal packings of equal prolate spheroids that will be discussed. The orientations are $a \parallel x$ in fig. 2 and $a \parallel y$ in fig. 3. An intermediate and a special orientation, both falling within the general class of "cubic" packing, are illustrated in fig. 4.

The fractional volume concentration C of solid spheres or prolate spheroids that form an assemblage systematic or otherwise, can be defined as

$$C = \frac{\text{Space occupied by solids}}{\text{Total space}}, \tag{4}$$

whilst the fractional porosity P becomes

$$P = (1 - C). \tag{5}$$

The same definitions apply to assemblages of irregular solids of more than one size.

Prolate spheroids in "cubic" packing occupy rows mutually at right angles, and the simplest modes of

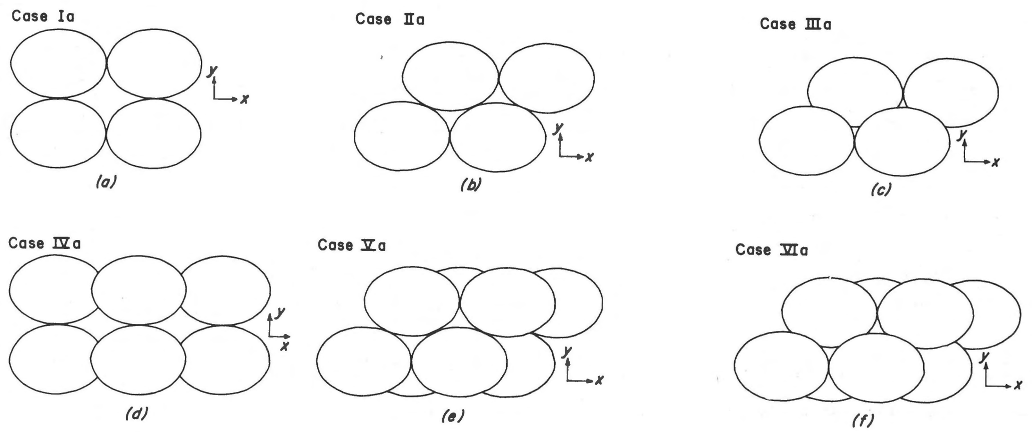

Fig. 2
The six systematic packings of equal prolate spheroids in the orientation $a \parallel x$.

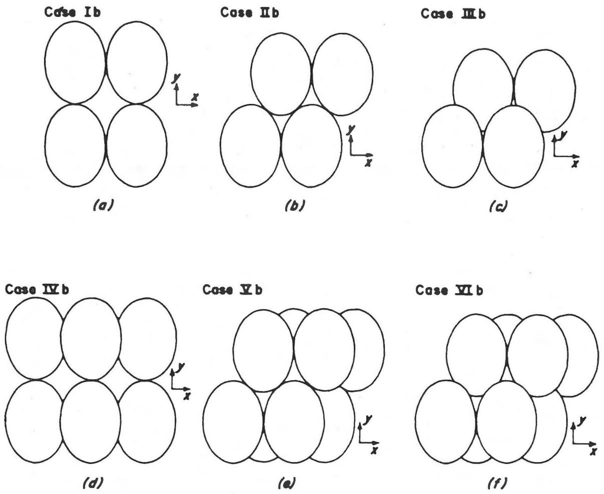

Fig. 3
The six systematic packings of equal prolate spheroids in the orientation $a \parallel y$.

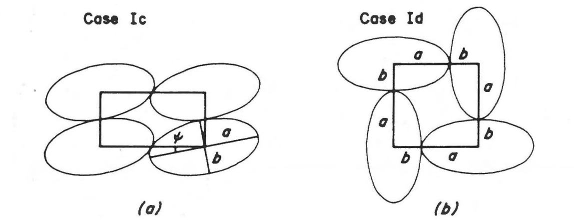

Fig. 4
Other 'cubic packings'. (a) Case I in the condition $0^\circ < \psi < 90^\circ$. (b) The 'card-house' arrangement.

such packing are shown in figs. 2a, 3a. In both cases Ia and Ib the fractional concentration is independent or whether $a \parallel x$ or $a \parallel y$ and is $(4/3\ \pi)/4\sqrt{4} = 0.524$, which is the same as the concentration of equal spheres in cubic packing (G r a t o n and F r a s e r, 1935). In the intermediate orientation (Case Ic), shown in fig. 4a, the spheroids again occur in rows mutually at right angles, but with their major axes uniformly inclined at an angle ψ from the x-axis. The fractional concentration now is

$$C = \frac{4/3\ \pi\, ab^2}{8\, bAB}\ , \tag{6}$$

in which

$$A = \sqrt{\frac{1}{\frac{1}{a^2} + \frac{\tan^2(90-\psi)}{b^2}}\ (1+\tan^2(90-\psi)}\ ,$$

and

$$B = \sqrt{\frac{1}{\frac{1}{a^2} + \frac{\tan^2\psi}{b^2}}\ (1+\tan^2\psi)}\ .$$

It will be seen from fig. 5 which is a plot of eq. (6) that the concentration is significantly affected by quite small tilts of the major axes of the spheroids.

The special or "card-house" mode of cubic packing (Case Id) illustrated in fig. 4b is characterised by the feature that for four of the spheroids $a \parallel x$ while for the remaining four $a \parallel y$. It will be noticed that this condition allows two different arrangements of the spheroids as grouped in vertical layers parallel to (x, y), and that in these layers the bodies lie at the corners of squares of side $(a + b)$. In either case the fractional concentration is easily seen to be

$$C = \frac{4/3\ \pi ab^2}{2b(a^2 + b^2)}\ , \tag{7}$$

whence C declines increasingly rapidly as b/a becomes smaller (fig. 6), on account of the "card-house" structure of the assemblages.

The "orthorhombic" packing of equal prolate spheroids is illustrated in two different orientations in fig. 2b, d and fig. 3b, d. In Cases IIa, b the spheroids in each layer lie at the corners of a rectangle, but in Cases IVa, b appear at the corners of a parallelogram. The fractional volume concentration is easily seen to be $(4/3\ \pi)/4\sqrt{3} = 0.605$, which is a constant independent of a or b and identical with the concentration of equal spheres in orthorhombic packing (G r a -

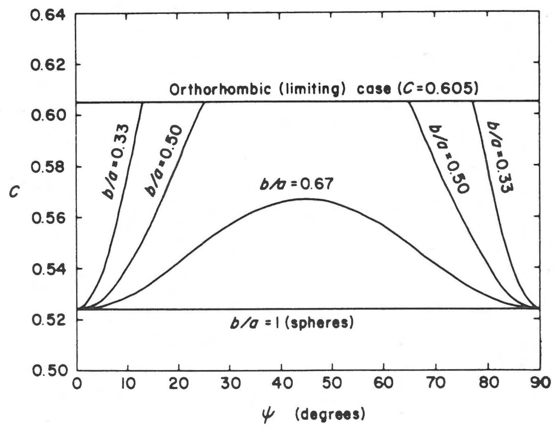

Fig. 5
Volume concentration as a function of angle of inclination of spheroid major axes for the 'cubic' packing of Case Ic.

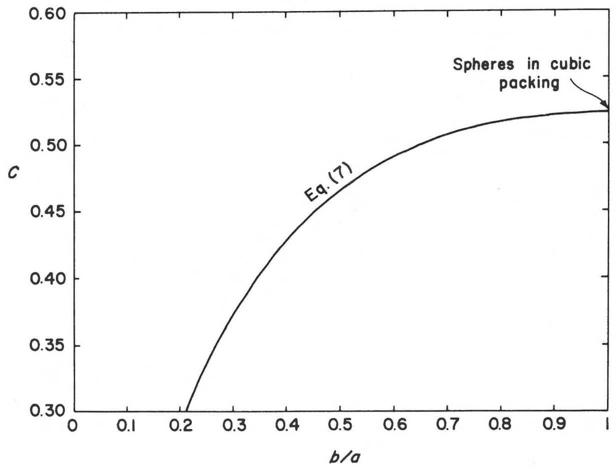

Fig. 6
Volume concentration as a function of axial ratio for the 'cubic' packing of Case Id.

ton and Fraser, 1935). It is also easily verified that, provided the spheroids remain tangent, the concentration in "orthorhombic" packing is not altered by inclining the major axes uniformly to the x-axis.

The "rhombohedral" packing of equal prolate spheroids (Cases IIIa, b; VIa, b) occurs in two different orientations, one based on the rectangular layer (fig. 2c, f) and the other on the parallelogram (fig. 3c, f). The fractional volume concentration becomes $(4/3\,\pi)/4\sqrt{2} = 0.741$, which is independent of a or b and is identical with the concentration of equal spheres in rhombohedral packing (Graton and Fraser, 1935). Again the concentration is independent of the uniform inclination of the major axes of the spheroids in the (x, y) plane, provided that tangency is retained.

There is only one orientation for an assemblage (Cases Va, b) of equal prolate spheroids in "tetragonal" packing (figs. 2e, 3e). When $a \parallel x$ (Case Va) the spheroids in each layer occur with their centres at the corners of a parallelogram. In the alternative orientation of the spheroids, when $a \parallel y$ (Case Vb), the centres lie at the corners of a rhomb. The fractional volume concentration of equal spheroids in "tetragonal" packing is $(4/3\,\pi)/6 = 0.698$, which is a constant independent of a and b and also of the inclination of the major axes in the (x, y) plane. Equal spheres in tetragonal packing have been shown by Graton and Fraser (1935) also to yield $C = 0.698$.

DILATANCY

When a mass of granular solids in close array experience a shearing force, the assemblage undergoes an increase of bulk volume which can also be measured as a decrease of the fractional concentration of solids or as an increase of the mean spacing of the particle centres. Osborne Reynolds (1885), to whom the prior investigations are due, called this phenomenon dilatancy.

We can write the concentration of equal spheres in an array in the form

$$\frac{1}{\lambda} = \frac{s}{D}\,, \tag{8}$$

where λ is the "linear concentration", s is the mean free separation distance between the spheres, and D is the sphere diameter (Bagnold, 1954). The linear concentration is related to the fractional volume concentration by the expression

$$\lambda = \frac{1}{(C_0/C)^{\frac{1}{3}} - 1}\,, \tag{9}$$

in which C_0 is the maximum possible concentration and C is the observed concentration. For equal spheres $C_0 = 0.741$ whence $\lambda = \infty$ when the spheres are packed rhombohedrally. Now the change in linear dimension of an array of equal spheres that experiences a shearing force can be written (Bagnold, 1967).

$$\delta = \frac{1}{\lambda_2} - \frac{1}{\lambda_1}\,, \tag{10}$$

where λ_2 is the concentration appropriate to the later state of the array and λ_1 is the concentration in the initial state. The change of linear dimension, δ, is the linear dilatation here in dimensionless form.

It will be evident that the above definitions and relationships are valid for random as well as systematic arrays of equal spheres, and also for arrays of spheres of unequal sizes to which a mean size can be ascribed. It will also be clear that they may be validly applied to random, i.e. isotropic, arrays of non-spherical particles of unequal sizes that can be described in terms of a suitable mean size. However, the mean size of non-spherical particles ceases to afford an unambiguous basis for these definitions and relationships once a dimensional anisotropy, such as one has in most natural sediments, appears in an array of the particles. For example, consider as in fig. 7 the

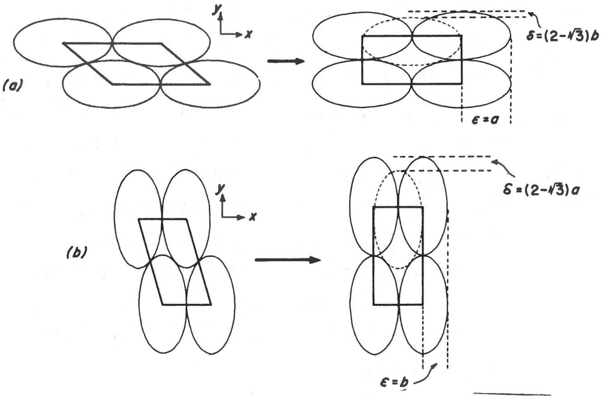

Fig. 7
Dilatant change of equal spheroids from an 'orthorhombic' to a 'cubic' packing.

dilatant change of a systematic array of equal prolate spheroids from the "orthorhombic" to the "cubic" packing (figs. 2, 3; Cases IIa→Ia, IIb→Ib). The change of volume concentration, $\Delta C = -(0.605\text{-}0.524) = -0.081$, is a constant independent of the orientation of the spheroids relative to the direction in which the dilatation is measured. The dimensional linear change or dilatation, on the other hand, is $\delta = (2-\sqrt{3})b$ when $a \parallel x$ but $\delta = (2-\sqrt{3})a$ when $a \parallel y$, and thus is a function of the orientation of the ellipsoids. A measure of λ consistent with C can only be obtained using an "effective diameter" $D' = 2b$ for the case when $a \parallel x$ and $D' = 2a$ when $a \parallel y$.

The dilatant behavior of granular solids is strikingly, and sometimes dangerously, evident in the sudden avalanching of material on scree slopes or in the lee of ripples and dunes. In these cases the dilatation, taking place normal to the plane of shearing, is sufficient to allow the particles making up the slope to pass freely over each other. Since fresh sediment is constantly being added to such slopes, with the effect of steepening them, an index of the stability of a slope subject to avalanching is afforded by the increase of slope-angle necessary to bring about a new avalanche. For it is well known that granular solids display two distinct slopes of respose: the angle of initial yield denoted by ϕ_i, and the smaller-valued angle of repose after shearing denoted by ϕ_r. The quantity

$$\Delta\phi = \phi_i - \phi_r \tag{11}$$

is the required index of slope stability, and may be called the dilatation angle.

Adapting an expression due to Bagnold (1967), we can write for the case of equal spheres

$$\Delta\phi = \tan^{-1} \frac{\dfrac{1}{\lambda_2} - \dfrac{1}{\lambda_1}}{\epsilon}, \tag{12}$$

where λ_1 is the linear concentration prior to shearing, λ_2 is the linear concentration at the completion of the dilatation, and ϵ is the tangential displacement of the spheres. This displacement, analogous to $1/\lambda$, is given by

$$\epsilon = \frac{d}{D}, \tag{13}$$

in which d, measured parallel to the surface of shearing, is the mean distance travelled by the spheres in order for the dilatation to be completed. Bagnold (1967) supposed that ϵ was of the order of unity, but this may be doubted as his definition is partly ambiguous. That ϵ is of the order of 0.5 seems more probable, even for random packings of unequal particles. In cases where ϵ is to be estimated for an assemblage of dimensionally oriented particles, it is clearly necessary to use an effective diameter D' determined by the orientation of the assemblage. Thus for prolate spheroids, and particles like them $\Delta\phi$ is strongly dependent on orientation when the assemblage is not a random one, since both δ and ϵ in their dimensional form depend on the particle orientation relative to the direction of shearing.

It is possible to calculate $\Delta\phi$ for particles of simple form arranged in simple ways. Fig. 8 shows some simple arrangements of prolate spheroids while Table 1 lists in these cases expressions for $\Delta\phi$. The arrangements are based on either a rectangular or a triangular layer of spheroids supposed rigidly held. In each case we consider the displacement, parallel to the x-axis and without change of orientation, of a spheroid placed in the hopper formed by the bodies in the rectangular or triangular layer below. The displacement in the case of the rectangular layer is either parallel to one side or to a diagonal of the rectangle. In the case of the triangular layer, it is parallel either to a side or to a median, there being two alternative directions of displacement in the latter instance. Also in each case we consider the orientations $a \parallel x$ and $a \parallel y$. The expressions for $\Delta\phi$ given in Table 1 are those which just allow the displaced spheroid to clear those below.

In fig. 9 the expressions for $\Delta\phi$ given in Table 1

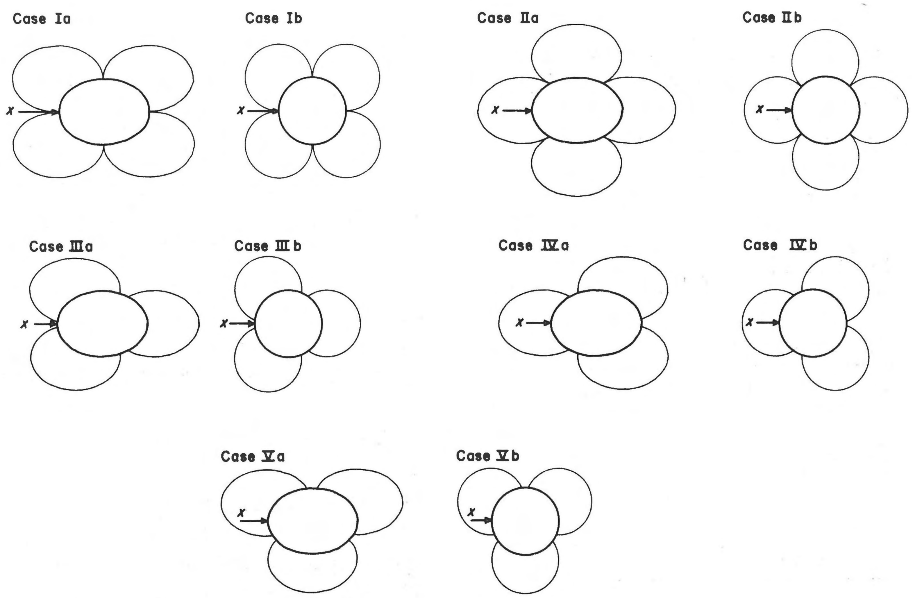

Fig. 8
Alternative modes of displacement of one spheroid relative to others rigidly held below.

are plotted against the axial ratic b/a. The curves sloping down to the right are for the orientation $a \parallel y$ while those sloping down to the left are for the orientation $a \parallel x$. The displacements based on the rectangular layer give the larger values of $\Delta \phi$, as is rather to be expected since the hopper formed within a rectangle of spheres or spheroids is deeper relatively than that within a triangle. It will be seen from the graph that $\Delta \phi$ is strongly influenced by the axial ratio as well as by the orientation of the spheroids relative to the direction of displacement. Broadly, the effect of increasing a to twice the value of b is to either halve or to double the value of $\Delta \phi$ from the case for spheres ($b/a = 1$), according to the orientation chosen for the major axes of the spheroids. Since a value of $b/a = 0.5$ is fairly typical of natural sand grains, and is on the high side for platy fragments in rock screes, it must be concluded that the shape and orientation of particles have a strong influence on the stability of the slopes that the particles underlie. In most cases the particles forming the slope rest with their long axes close to the direction of the slope. Such surfaces would be highly unstable under conditions when fresh particles can be constantly added.

It will be evident that when the spheroids lie in an orientation such that $0° < \psi < 90°$, the dilatation angle for each packing lies between the limits indicated in fig. 9.

In the developments which led to fig. 9 we assumed that the free spheroid was displaced parallel to the horizontal. Now it is clear that if the displacement takes place in the gravity field on a slope of declination β, where $\beta \neq 0°$, then the observed value of the dilatation angle $\Delta \phi$ as defined by eq. (11), may fall short of the theoretical value by

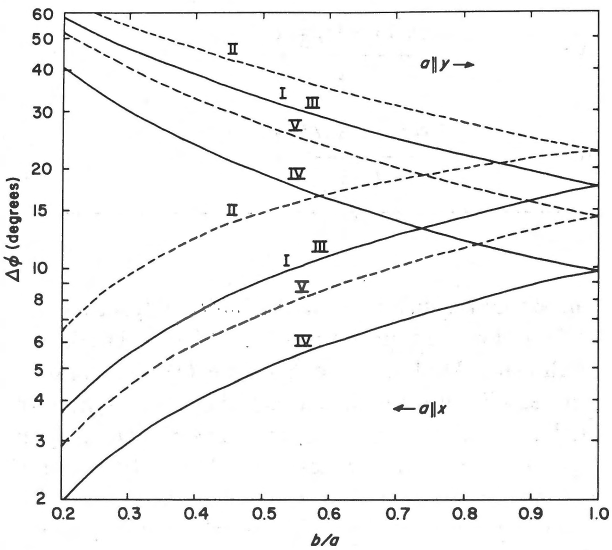

Fig. 9
Dilatation angle as a function of axial ratio for the modes of displacement shown in fig. 8.

TABLE 1

Expressions for the dilatation angle and maximum angle of initial yield for the spheroid arrangements shown in fig. 8.

Case	$\tan \Delta\phi$	$\tan \phi_{i(max)}$
Ia	$\dfrac{(\sqrt{3}-\sqrt{2})\,b}{a}$	$\dfrac{a}{\sqrt{2}\,b}$
Ib	$\dfrac{(\sqrt{3}-\sqrt{2})\,a}{b}$	$\dfrac{b}{\sqrt{2}\,a}$
IIa	$\dfrac{(2-\sqrt{2})\,b}{\sqrt{2}\sqrt{\frac{1}{2}(a^2+b^2)}}$	$\dfrac{\sqrt{\frac{1}{2}(a^2+b^2)}}{b}$
IIb	$\dfrac{(2-\sqrt{2})\,a}{\sqrt{2}\,b}$	$\dfrac{b}{a}$
IIIa	$\dfrac{(2-2\sqrt{\frac{2}{3}})\,b}{2/\sqrt{3}\,a}$	$\dfrac{a}{\sqrt{2}\,b}$
IIIb	$\dfrac{(2-2\sqrt{\frac{2}{3}})\,a}{2/\sqrt{3}\,b}$	$\dfrac{b}{\sqrt{2}\,a}$
IVa	$\dfrac{(\sqrt{3}-2\sqrt{\frac{2}{3}})\,b}{1/\sqrt{3}\,a}$	$\dfrac{1/\sqrt{3}\,a}{2\sqrt{\frac{2}{3}}\,b}$
IVb	$\dfrac{(\sqrt{3}-2\sqrt{\frac{2}{3}})\,a}{1/\sqrt{3}\,b}$	$\dfrac{1/\sqrt{3}\,b}{2\sqrt{\frac{2}{3}}\,a}$
Va	$\dfrac{(\sqrt{13}-4\sqrt{\frac{2}{3}})\,b}{2/3\,a}$	–
Vb	$\dfrac{(\sqrt{13}-4\sqrt{\frac{2}{3}})\,a}{2/3\,b}$	–

an amount dependent on the limit to ϕ_i for a given orientation and packing set by the declination β. Following Miller and Byrne (1966), consider the equilibrium in the vertical plane of a sphere of radius r set in the hopper between two tangent spheres of the same radius r (fig. 10). Let the line of centres of the two lower spheres be inclined parallel to a slope of declination β. The sphere resting in the hopper will be in limiting equilibrium when the line joining its centre with the centre of the sphere below becomes vertical. We then have

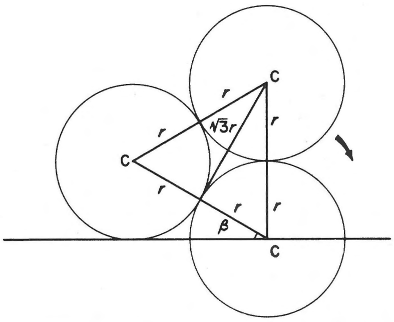

Fig. 10
Limiting equilibrium of a sphere contained in the hopper formed by two tangent spheres of the same radius.

$$\beta_{crit} = \tan^{-1}\frac{1}{\sqrt{3}} = \phi_{i(max)}\,, \qquad (14)$$

in which $\phi_{i(max)}$ is the steepest angle which can theoretically be assumed by a slope underlain by spheres packed and oriented in the manner of fig. 10. The observed value of the dilatation angle for a slope under active construction therefore cannot exceed

$$\Delta\phi = \phi_{i(max)} - \phi_r\,, \qquad (15)$$

where $\phi_{i(max)}$ is appropriate to the sphere assemblage under consideration. The above ideas can be readily extended to the spheroid packings of fig. 8, whence the expressions for $\phi_{i(max)}$ listed in Table 1 are obtained. Values of $\phi_{i(max)}$ are given in fig. 11 for a range of value of b/a. Also included in the figure is a curve for spheroids in the packing analogous to that of spheres in fig. 10. Those curves sloping down to the right are for the orientation $a \parallel x$ while those sloping down to the left are for the orientation $a \parallel y$. It is assumed in all cases that there is no slipping between the spheroids until the appropriate line of centres becomes vertical. For real spheroids there is an upper limit of the angle of yield imposed by slip between the spheriods. The existence of this further limit on slope is likely to level out the upper parts of the branches of the curves sloping to the right.

It would appear theoretically that the maximum slope to which spheroids can be built depends on the concentration of spheroids, their shape, and their orientation relative to the direction of slope. For a given concentration and packing orientation, steeper slopes can be obtained when the major axes of spheroids lie parallel to the slope than when the major axes are perpendicular to the slope. It is easily verified that for orientations intermediate between parallel and perpendicular to slope, the maximum angle of initial yield takes values that lie between the

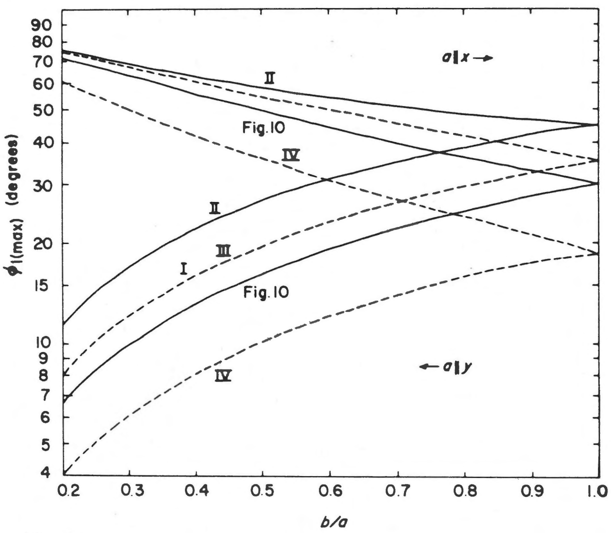

Fig. 11
Maximum angle of initial yield of the spheroid arrangements in fig. 8 as a function of axial ratio.

limiting curves shown in fig. 11. The theoretical maximum angle of initial yield does, however, depend on the orientation of the packing relative to the direction of sliding. In rhombohedral packing, for example, sliding may take place either parallel to the median of a triangular arrangement of solids, or parallel to an edge or a diagonal of solids arranged in a rectangular layer. For spheres the values of $\phi_{i(max)}$ for these orientations of the one packing cover the rather broad range from 18°30′ to 45°00′. The mean of $\phi_{i(max)}$ for the three orientations of rhombohedral packing is approximately 33°. Spheres in cubic packing are unstable at all values of β, whence $\phi_{i(max)} = 0°$.

The angle of initial yield of assemblages of equal spheres should therefore depend on concentration and should vary between a value in the range 18°-45° for closest systematic packing and 0° for loosest systematic packing. It is significant that in experiments on narrowly graded glass beads, $\phi_i \approx 39°$ for closest random packing ($C \approx 0.64$) but falls to $\phi_i \approx 25°$ for a looser random packing ($C \approx 0.60$). Natural sands, whose particles are ellipsoidal, should assume values of ϕ_i considerably in excess of the 33° for spheres, when tightly packed on a slope in such a manner that the major axes are in subparallel alignment with the slope. Precisely this is observed. It is also observed that ϕ_i for natural sands in loose packing is in the neighbourhood of 32°, which is some 7° larger than the value for loose-packed spheres.

SUMMARY OF CONCLUSIONS

Because the majority of real sedimentary particles resemble in form ellipsoids rather than spheres, the use of the sphere as the ideal particle has serious drawbacks when it is desired to make theoretical models of the properties of particle assemblages that depend on particle shape, packing and orientation. In the present investigation, therefore, the prolate spheroid as the simplest type of ellipsoid has been used as the ideal sedimentary particle. Especial emphasis has been placed in the investigation on the packing and dilatation of assemblages of prolate spheroids. The main conclusions can be summarised as follows.

Equal prolate spheroids can be systematically packed in six ways analogous to the six packings of equal spheres. However, the spheroid packings can assume different orientations depending on the orientation of the constituent spheroids.

The volume concentration of spheroids in a systematic packing is identical with the volume concentration of spheres in the packing with which analogy is drawn.

The volume concentration of spheroids in a systematic packing is independent of spheroid orientation in all packings except the "cubic". There is one special "cubic" packing of equal spheroids in which the volume concentration is a function of the axial ratio. In this packing the spheroids have a "card-house" arrangement.

The dilatation angle of equal spheroids is a function of the manner and orientation of the packing and of the spheroid orientation relative to the direction of displacement.

The theoretical angle of initial yield of assemblages of equal spheroids is dependent on the manner and orientation of packing and on the spheroid orientation relative to the direction of yield. The maximum angle of yield is directly proportional to volume concentration.

Since the sphere is a special ellipsoid, the model based on the spheroid has properties in common with the model using the sphere. However, the quantitative differences between the two models are significant at the axial ratios relevant to natural sedimentary particles. We are therefore justified in using a model based on the spheroid in order to obtain a better theoretical understanding of the properties of detrital sediments.

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